

Matrix Strings in Weakly Curved Background Fields

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Abstract

We investigate further the recent proposal for the form of the Matrix theory action in weak background fields. We perform DVV reduction to the multiple $D0$ -brane action in order to find the Matrix string theory action for multiple fundamental strings in curved but weak NS-NS and R-R backgrounds. This matrix sigma model gives a definite prescription on how to deal with R-R fields with an explicit spacetime dependence in Type II string theory. We do this both via the 9 – 11 flip and the chain of T and S dualities, and further check on their equivalence explicitly. In order to do so, we also discuss the implementation of S -duality in the operators of the 2-dimensional world-volume supersymmetric gauge theory describing the Type IIB D -string. We compare the result to the known Green-Schwarz sigma model action (for one string), and use this comparison in order to discuss about possible, non-linear background curvature corrections to the Matrix string action (involving many strings), and therefore to the Matrix theory action. We illustrate the nonabelian character of our action with an example involving multiple fundamental strings in a non-trivial R-R flux, where the strings are polarized into a noncommutative configuration. This corresponds to a dielectric type of effect on fundamental strings.

April 2000

1 Introduction

The five known superstring theories as well as the low-energy 11-dimensional supergravity are known to be related through a web of dualities, and it is believed that all these theories are simply different limits of an underlying 11-dimensional quantum theory known as M -theory, whose fundamental degrees of freedom are as yet unknown, but that can be defined as the strong coupling limit of Type IIA string theory [1, 2]. Let us first recall that M -theory compactified on a circle is described by Type IIA string theory at finite string coupling. It is by now a well known conjecture that M -theory compactified on a lightlike circle admits a nonperturbative description in terms of the degrees of freedom of a collection of $D0$ -branes [3, 4, 5, 6, 7, 8].

Matrix theory encodes a great deal of information about the structure of both M -theory and 11-dimensional supergravity (some reviews are [9, 10, 11]). One knows how to identify supergravitons, membranes and fivebranes in Matrix theory [4, 5, 12], and the interactions between these objects in Matrix theory have been found to agree with supergravity in a variety of situations. In particular, for general Matrix configurations, it was shown in [13, 14] that the supergravity potential between an arbitrary pair of M -theory objects arising from the exchange of quanta with zero longitudinal momentum is exactly reproduced by terms in the one-loop Matrix theory potential. These results were also used to describe a formulation of Matrix theory in a general metric and 3-form background, via a matrix sigma model type of action [14]. Such matrix sigma model actions had also been advocated for earlier in [15]. A different type of approach to the 3-form background is, *e.g.* [16].

A question that naturally arises is that if we have a formulation of Matrix theory in curved background fields, that should somehow yield a matrix formulation of Type II string theory in curved background fields, and in particular in the presence of R-R fields. Moreover, due to the second quantized nature of the Matrix theory formalism, we should be able to obtain in this way a description of multiple interacting strings in both NS-NS and R-R curved backgrounds. This would be quite interesting, as even for a single fundamental superstring the action in a general background including arbitrary R-R fields is not yet well understood.

Due to the relation between Type IIA string theory and M -theory, it is possible to construct a matrix theory formulation of superstring theory which is known as matrix string theory [17, 18, 19, 20, 21]. Such a formulation is achieved once one understands toroidal compactifications of Matrix theory [22], for then the particular case of the S^1 compactification will lead to the matrix formulation of the Type IIA superstring – as M -theory compactified on a circle yields the IIA theory, where the IIA string is obtained from the wrapped $M2$ -brane [23]. Recall that this matrix string theory is a supersymmetric gauge theory that not only contains all of the DLCQ IIA superstring theory, but also contains extra degrees of freedom which represent nonperturbative objects in string theory. These nonperturbative degrees of freedom represent the inclusion of D -brane states, and also give us a prescription to include nonperturbative corrections in calculations of diverse processes in perturbative string theory.

Because we know a great deal about Type IIA string theory, matrix string theory is a very good laboratory to test Matrix theory. Of course ideally we would like to have a microscopic definition of M -theory which would be covariant and defined in arbitrarily curved backgrounds. But due to the nonabelian character of the theory such is not an easy goal. Information from the abelian limit of the theory may then prove to be of great value in trying to deal with such issues, and a precious source of information on this abelian limit is undoubtedly the Type II theory. In flat space the matrix string theory action has been lifted from the cylinder to its branched coverings and a precise connection with the Green–Schwarz action in light-cone gauge has been achieved [24, 25, 26, 27, 28, 29], with the interesting result that the full moduli space of the IIA theory is recovered within matrix string theory only in the large N limit. Scattering amplitudes have been reproduced within the matrix string formalism in [30, 31], for reviews on several issues see [21, 32]. More recently, the issue of a spacetime covariant formulation of matrix string theory has been addressed in [33], but this is a matter which is far from settled.

This paper concerns the generalization of matrix string theory when in the presence of weakly curved background fields. In particular, we want to address the question of how to describe multiple interacting strings in NS–NS and R–R curved backgrounds. Indeed, because it is known how to describe the linear couplings of Matrix theory to a curved 11-dimensional background, we shall also be able to find the linear couplings of matrix string theory to a curved 10-dimensional background. This could be of great interest not only in trying to improve our knowledge of string theory in R–R backgrounds, but also when comparing to the IIA theory in the abelian limit we could expect for new information on how to construct Matrix theory in a general curved background. In summary, we are looking for a matrix string sigma model type of action,

$$\begin{aligned} \mathcal{S} = & \frac{1}{2\pi} \int d\sigma d\tau \left(\frac{1}{2} g_{\mu\nu}^{IIA}(X) I_g^{\mu\nu} + \phi(X) I_\phi + B_{\mu\nu}(X) I_s^{\mu\nu} + \tilde{B}_{\mu\nu\lambda\rho\sigma\tau}(X) I_5^{\mu\nu\lambda\rho\sigma\tau} \right. \\ & \left. + C_\mu(X) I_0^\mu + \tilde{C}_{\mu\nu\lambda\rho\sigma\tau\xi}(X) I_6^{\mu\nu\lambda\rho\sigma\tau\xi} + C_{\mu\nu\lambda}(X) I_2^{\mu\nu\lambda} + \tilde{C}_{\mu\nu\lambda\rho\sigma}(X) I_4^{\mu\nu\lambda\rho\sigma} \right), \end{aligned} \quad (1)$$

and we shall precisely explain in this paper how to construct this action by specifying both the I tensor couplings as well as the inclusion of spacetime dependence in the (weak) background fields. The explicit form of all these tensor couplings is presented in section 4.2.

We shall begin in section 2 with a brief review of the work done for the case of Matrix theory in weak background fields [13, 14, 34, 35]. We shall recall that there is a definite proposal on how to supplement the flat space Matrix action with linear couplings between the background fields – the supergraviton, the membrane and the fivebrane – and the respective Matrix descriptions for the supergravity stress–energy tensor, membrane current and fivebrane current. Moreover we shall also recall that through the Sen–Seiberg limiting procedure, this action can be reduced to an action for multiple $D0$ -branes in weakly curved Type IIA background fields. By T -duality this can be extended to any Type II D -brane. Then, in section 3 we present a brief review of the Dijkgraaf–Verlinde–Verlinde (DVV) re-

duction of Matrix theory to matrix string theory [19], via both the so-called 9 – 11 flip and also the T – S – T chain of dualities. This will be of fundamental use in the sections that follows, as we shall be generalizing that procedure to the curved background situation.

In the following sections we perform the DVV reduction to the multiple $D0$ –brane action in order to find the matrix string theory action for multiple fundamental strings in curved but weak NS–NS and R–R backgrounds. As we just said, this is a generalization of the work by DVV. These sections deal with a great deal of algebra, and we will be schematic in presenting our results. The matrix sigma model obtained in this way gives a definite prescription on how to deal with R–R fields with an explicit spacetime dependence in Type II string theory. Due to the nonabelian nature of the action, it also gives a second quantized description of Type II string theory in such backgrounds. We shall obtain the matrix string sigma model both via the 9 – 11 flip (described in section 4) and the chain of T and S dualities (described in section 5), and further check their equivalence explicitly by obtaining the same results in both cases. In order to do so, we will need to discuss in section 5 the implementation of S –duality in the composite operators of the 2–dimensional world–volume supersymmetric gauge theory describing the Type IIB D –string. We shall obtain the S –duality transformations for the world–volume fields from the equivalence with the 9 – 11 flip, and we shall see that these transformation properties are indeed quite simple, as should be expected.

In section 6 we compare the result to the known Green–Schwarz sigma model action (for one string) [36]. This is done by extracting the free string limit (the IR limit of the gauge theory) of the matrix string theory action. This will be a qualitative match only, as we shall not construct the precise lifting of the matrix string action to the Green–Schwarz action. We then use this comparison in order to discuss about possible, non–linear background curvature corrections to the matrix string action (involving many strings), and therefore to the Matrix theory action. Again this is a qualitative analysis, but it gives us further insight into the goal of constructing Matrix theory in arbitrary curved backgrounds. Then, in section 7, we briefly discuss the exponentiation of the noncommutative vertex operators we obtained in order to build coherent states of fundamental strings and so obtain the full non–linear matrix string sigma model. As such a construction is not clear at this stage, we turn to an illustration of the nonabelian character of our action with an example, namely multiple fundamental strings in a non–trivial R–R flux, where the strings are polarized into nonabelian configurations due to the background field. This means that Myers’ dielectric effect for D –branes has an analogue for fundamental strings. We also speculate on a possible relation between this effect and string theory noncommutative background geometries, where this could provide a very interesting example of target space noncommutativity in the presence of R–R fields (as opposed to recent discussions of world–volume noncommutativity in the presence of NS–NS fields, *e.g.*, [10, 37, 38, 39, 40]). We conclude in section 8 with some open problems for future research.

2 Matrix Theory in Weakly Curved Backgrounds

We begin with a short review of the results obtained for Matrix theory in weakly curved background fields [13, 14], and also for the action of multiple $D0$ -branes in weak Type IIA backgrounds [34] as well as for the action of multiple Dp -branes in Type II weak background fields [35].

2.1 Results for Matrix Theory

In this section we briefly review the results in [13, 14] dealing with the construction of a Matrix theory action in weak M -theory backgrounds. As we shall see, due to the 9 – 11 flip in the DVV construction of matrix string theory, we will have particular interest in the tensors that appear in this Matrix theory action.

The proposal in question actually concerns the terms in the action of Matrix theory which are linear in the background fields [14]. If we consider a general Matrix theory background, with metric $g_{IJ} = \eta_{IJ} + h_{IJ}$ and 3-form field A_{IJK} , then the linear effects of this background can be described by supplementing the flat space Matrix theory action,

$$S_{Flat} = \frac{1}{R} \int dt \, \text{Tr} \left(\frac{1}{2} D_t X^i D_t X^i - \frac{1}{2} \sum_{i < j} [X^i, X^j]^2 - \frac{1}{2} \Theta D_t \Theta + \frac{1}{2} \Theta \gamma^i [X_i, \Theta] \right), \quad (2)$$

with additional linear coupling terms of the form,

$$S_{Weak} = \int dt \sum_{n=0}^{\infty} \sum_{i_1, \dots, i_n} \frac{1}{n!} \left\{ \frac{1}{2} T^{IJ(i_1 \dots i_n)} \partial_{i_1} \dots \partial_{i_n} h_{IJ}(0) + J^{IJK(i_1 \dots i_n)} \partial_{i_1} \dots \partial_{i_n} A_{IJK}(0) \right. \\ \left. + M^{IJKLMN(i_1 \dots i_n)} \partial_{i_1} \dots \partial_{i_n} \tilde{A}_{IJKLMN}(0) + \text{Fermionic Terms} \right\}, \quad (3)$$

where \tilde{A} is the dual 6-form field which satisfies at linear order,

$$d\tilde{A} = \star dA. \quad (4)$$

The previous matrix expressions $T^{IJ(i_1 \dots i_n)}$, $J^{IJK(i_1 \dots i_n)}$ and $M^{IJKLMN(i_1 \dots i_n)}$ are the Matrix theory forms of the multipole moments of the stress-energy tensor, membrane current and 5-brane current of 11-dimensional supergravity. Explicit forms for the bosonic parts of these moments were first given in [13], and those results were later extended to quadratic fermionic terms (and also some quartic fermionic terms) in [14]. The complete results in [13, 14] are reproduced in the Appendix.

With these definitions the previous expressions yield a formulation of Matrix theory in a weak background metric to first order in h_{IJ} , the 3-form A_{IJK} , and all their higher derivatives. It was moreover argued in [14] that if the Matrix theory conjecture is true in flat space, then this formulation must be correct at least to order $\mathcal{O}(\partial^4 h, \partial^4 A)$. It was

also conjectured in that paper that this form may work to all orders in derivatives of the background fields, and in a general background. One should observe however that it is not known how to incorporate dependence of the background on the compact coordinate X^- .

2.2 Results for Multiple D-branes

We proceed by reviewing how the previous results can be used to construct actions for multiple $D0$ -branes [34] and in general for multiple Dp -branes [35] in Type II string theory, in the approximation of weak background fields. Of particular interest to our goal in this paper is the case of the $D0$ -brane action, due to the duality sequence in the DVV construction of matrix string theory and its associated 9 – 11 flip.

To start, we shall recall from [34] how one obtains the action for multiple $D0$ -branes in background fields, as this will later prove its interest when we try to do the same for the matrix string action. We begin with M -theory on a background metric, $g_{IJ} = \eta_{IJ} + h_{IJ}$, in a frame where there is a compact coordinate X^- of size R , which becomes lightlike in the flat space limit, $g_{IJ} \rightarrow \eta_{IJ}$. From the Sen–Seiberg limit [7, 8] we know that this theory can be described as a limit of a family of spacelike compactified theories. If we define an \tilde{M} -theory with background metric $\tilde{g}_{IJ} = \eta_{IJ} + \tilde{h}_{IJ}$, in a frame with a spacelike compact coordinate X^{11} of size R_{11} , then the DLCQ limit of the original M -theory is found by boosting the \tilde{M} -theory along X^{11} , and then taking the limit $R_{11} \rightarrow 0$. Knowing the boost we can trivially Lorentz relate the metric \tilde{g}_{IJ} in the \tilde{M} -theory with the metric g_{IJ} in the M -theory. Moreover, in the DLCQ description the M -theory is in light-cone coordinates, $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{11})$, and so it is easy to relate the metric \tilde{g}_{IJ} to the light-cone metric g_{IJ} .

Of course our final goal is more than we have just obtained. We would like to relate the Type IIA string theory background fields to the DLCQ M -theory ones. But this is now straightforward. \tilde{M} -theory on a small spacelike circle of radius R_{11} is known to be equivalent to Type IIA string theory with background fields given to leading order by,

$$\begin{aligned} h_{\mu\nu}^{IIA} &= \tilde{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\tilde{h}_{11\,11}, \\ C_\mu &= \tilde{h}_{11\,\mu}, \\ \phi &= \frac{3}{4}\tilde{h}_{11\,11}. \end{aligned} \tag{5}$$

All we have left to do is to relate the \tilde{h}_{IJ} metric to the h_{IJ} one through the previously explained procedure. In order to describe nontrivial background antisymmetric tensor fields, one should also include the connections between the IIA background fields and the M -theory background 3-form field. The action for multiple $D0$ -branes can now be obtained by direct comparison with the one for Matrix theory just described in the previous subsection. Indeed [34], one can first write the $D0$ -brane action in terms of some unknown quantities coupling to the background fields. These quantities will be denoted by I_x and will couple linearly to

each of the background fields, so that to leading order the action for N $D0$ -branes is written as:

$$\begin{aligned}
S_{D0-branes} = S_{Flat} &+ \int dt \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \frac{1}{2} (\partial_{k_1} \cdots \partial_{k_n} h_{\mu\nu}^{IIA}) I_h^{\mu\nu(k_1 \cdots k_n)} + (\partial_{k_1} \cdots \partial_{k_n} \phi) I_{\phi}^{(k_1 \cdots k_n)} \right. \\
&+ (\partial_{k_1} \cdots \partial_{k_n} C_{\mu}) I_0^{\mu(k_1 \cdots k_n)} + (\partial_{k_1} \cdots \partial_{k_n} \tilde{C}_{\mu\nu\lambda\rho\sigma\tau\xi}) I_6^{\mu\nu\lambda\rho\sigma\tau\xi(k_1 \cdots k_n)} \\
&+ (\partial_{k_1} \cdots \partial_{k_n} B_{\mu\nu}) I_s^{\mu\nu(k_1 \cdots k_n)} + (\partial_{k_1} \cdots \partial_{k_n} \tilde{B}_{\mu\nu\lambda\rho\sigma\tau}) I_5^{\mu\nu\lambda\rho\sigma\tau(k_1 \cdots k_n)} \\
&\left. + (\partial_{k_1} \cdots \partial_{k_n} C_{\mu\nu\lambda}) I_2^{\mu\nu\lambda(k_1 \cdots k_n)} + (\partial_{k_1} \cdots \partial_{k_n} \tilde{C}_{\mu\nu\lambda\rho\sigma}) I_4^{\mu\nu\lambda\rho\sigma(k_1 \cdots k_n)} \right\}. \quad (6)
\end{aligned}$$

Replacing in this action the background fields of the Type IIA string theory by the background fields of DLCQ M -theory according to the previous relations, one can then compare the previous action for $D0$ -branes to the Matrix theory action and deduce the expressions for the string theory couplings I_x . These are [34]:

$$\begin{aligned}
I_h^{00} &= T^{++} + T^{+-} + (I_h^{00})_8 + \mathcal{O}(X^{12}), \\
I_h^{0i} &= T^{+i} + T^{-i} + \mathcal{O}(X^{10}), \\
I_h^{ij} &= T^{ij} + (I_h^{ij})_8 + \mathcal{O}(X^{12}), \\
I_{\phi} &= T^{++} - \frac{1}{3}(T^{+-} + T^{ii}) + (I_{\phi})_8 + \mathcal{O}(X^{12}), \\
I_s^{0i} &= 3J^{+-i} + \mathcal{O}(X^8), \\
I_s^{ij} &= 3J^{+ij} - 3J^{-ij} + \mathcal{O}(X^{10}), \\
I_0^0 &= T^{++}, \\
I_0^i &= T^{+i}, \\
I_2^{0ij} &= J^{+ij} + \mathcal{O}(X^{10}), \\
I_2^{ijk} &= J^{ijk} + \mathcal{O}(X^8), \\
I_4^{0ijkl} &= 6M^{+-ijkl} + \mathcal{O}(X^8), \\
I_4^{ijklm} &= -6M^{-ijklm} + \mathcal{O}(X^{10}), \\
I_6^{0ijklmn} &= S^{+ijklmn} + \mathcal{O}(X^{10}), \\
I_6^{ijklmnp} &= S^{ijklmnp} + \mathcal{O}(X^{12}). \quad (7)
\end{aligned}$$

By T -duality of background supergravity fields and T -duality of world-volume fields, the previous action for N $D0$ -branes can be transformed into an action for N Type II Dp -branes, as was discussed in [35]. This also allows for a discussion of nonabelian terms in the Born-Infeld action. For further discussion we refer the reader to the original references [34, 35].

3 Matrix String Theory

According to the DVV formulation of matrix string theory [19], one can reduce the Matrix theory action to an action for IIA matrix strings in two different ways. One way is by performing the so-called 9–11 flip, where one exchanges the role of the 9th and 11th directions of M -theory. Another way is via a set of dualities on the background fields. Moreover, the coordinate flip should clearly be equivalent to this specific chain of dualities. In here, one starts by T -dualizing and then takes an S -duality followed by another T -duality. The starting point is the Type IIA theory, with N_{11} yielding the D -particle number. After the T -duality along R_9^{IIA} one reaches Type IIB, where N_{11} now equals the D -string number. The Type IIB S -duality leads to N_{11} equaling the F -string number, and the final T -duality along R_9^{IIB} leads back to Type IIA, with N_{11} now being equal to the F -string momenta.

In order to see that this exactly matches the simple 9–11 flip on the compact coordinates, let us follow these dualities with a slightly greater detail [19]. If we compactify M -theory on $\mathbf{S}_{R_9}^1 \times \mathbf{S}_{R_{11}}^1$, with R_{11} the spacelike compact direction which becomes lightlike in the Sen–Seiberg limit, we will have the parameters,

$$R_{11} = g_s \ell_s, \quad \ell_P^3 = g_s \ell_s^3, \quad (8)$$

and also R_9 for the remaining spacelike compact direction. The 9 – 11 flip simply leads to the IIA theory with parameters,

$$R_9 = g'_s \ell_s, \quad \ell_P^3 = g'_s \ell_s^3, \quad (9)$$

where now the remaining spacelike compact direction is R_{11} . On the other hand, given our starting point and T -dualizing along R_9^{IIA} , one obtains the following Type IIB parameters,

$$g_s^{IIB} = \frac{\ell_s}{R_9^{IIA}} g_s^{IIA} = \frac{\ell_s}{R_9^{IIA}} \frac{R_{11}}{\ell_s} = \frac{R_{11}}{R_9^{IIA}}, \quad (10)$$

$$R_9^{IIB} = \frac{\alpha'}{R_9^{IIA}} = \frac{\ell_s^2}{R_9^{IIA}}. \quad (11)$$

A further IIB S -duality leads to

$$g'_s{}^{IIB} = \frac{1}{g_s^{IIB}} = \frac{R_9^{IIA}}{R_{11}}, \quad (12)$$

$$R'_9{}^{IIB} = \frac{1}{g_s^{IIB}} R_9^{IIB} = \left(\frac{R_{11}}{R_9^{IIA}} \right)^{-1} \frac{\ell_s^2}{R_9^{IIA}}. \quad (13)$$

In the expressions above for the radius, recall that under T -duality it is the Einstein frame metric that is invariant. The string frame metric gets transformed with a g_s factor. Finally, we finish the chain of dualities by T -dualizing back to the IIA theory along R_9^{IIB} . We end up with the parameters,

$$g_s'^{IIA} = \frac{\ell_s}{R_9'^{IIB}} g_s'^{IIB} = \frac{R_9}{\ell_s}, \quad (14)$$

$$R_9'^{IIA} = \frac{\ell_s^2}{R_9'^{IIB}} = R_{11}, \quad (15)$$

which are exactly the same as the ones obtained via the 9 – 11 flip.

Given that, as we have just seen, the chain of dualities is equivalent to the 9 – 11 flip, we shall now obtain the matrix string action from the Matrix action by following the most straightforward path, *i.e.*, we shall simply perform the flip to the Matrix theory action [19]. With the dimensionfull parameters made explicit, in order to produce the correct dimensions for the fields, the Matrix action in a flat background is written as,

$$S = \int dt \, \text{Tr} \left(\frac{1}{2R} \dot{X}_i \dot{X}_i + \frac{RM_P^6}{8\pi^2} \sum_{i < j} [X^i, X^j]^2 + \frac{iM_P^3}{4\pi} \theta^T \dot{\theta} - \frac{RM_P^6}{8\pi^2} \theta^T \gamma_i [X^i, \theta] \right), \quad (16)$$

with $R = 2\pi\ell_P^3$ and M_P is the Planck mass.

We further consider the theory compactified along the 9^{th} direction. Therefore, defining $\hat{R}_9 = \frac{\alpha'}{R_9}$, one T -dualizes according to the standard procedure and obtains:

$$\begin{aligned} S' = & \int dt \frac{1}{2\pi\hat{R}_9} \int_0^{2\pi\hat{R}_9} d\hat{x} \, \text{Tr} \left(\frac{1}{2R} \dot{X}_i \dot{X}_i + \frac{1}{2R} (2\pi\alpha')^2 \dot{A}^2 + \frac{RM_P^6}{8\pi^2} \sum_{i < j} [X^i, X^j]^2 \right. \\ & \left. - \frac{RM_P^6}{8\pi^2} (2\pi\alpha')^2 (D_{\hat{x}} X^i)^2 + \frac{iM_P^3}{4\pi} \theta^T \dot{\theta} - \frac{RM_P^6}{8\pi^2} \theta^T \gamma_i [X^i, \theta] - i \frac{RM_P^6}{8\pi^2} (2\pi\alpha') \theta^T \gamma_9 D_{\hat{x}} \theta \right). \end{aligned}$$

The implementation of the 9 – 11 flip is quite simple, as one just has to notice the change in parameters so that $R_9 = g_s \ell_s$ and $\hat{R}_9 = \frac{\ell_s}{g_s}$. Consequently,

$$\begin{aligned} S' = & \int dt \frac{g_s}{2\pi\ell_s} \int_0^{2\pi\frac{\ell_s}{g_s}} d\hat{x} \, \text{Tr} \left(\frac{1}{2R} \dot{X}_i \dot{X}_i + \frac{2\pi^2\ell_s^4}{R} \dot{A}^2 + \frac{RM_P^6}{8\pi^2} \sum_{i < j} [X^i, X^j]^2 \right. \\ & \left. - \frac{1}{2} RM_P^6 \ell_s^4 (D_{\hat{x}} X^i)^2 + \frac{iM_P^3}{4\pi} \theta^T \dot{\theta} - \frac{RM_P^6}{8\pi^2} \theta^T \gamma_i [X^i, \theta] - \frac{iRM_P^6 \ell_s^2}{4\pi} \theta^T \gamma_9 D_{\hat{x}} \theta \right). \end{aligned}$$

One can rescale the world-sheet coordinates, from (\hat{x}, t) to (σ, τ) , such that $0 < \sigma < 2\pi$ and so that the coordinates on the cylinder become dimensionless. For that one changes $\hat{x} = \frac{\ell_s}{g_s} \sigma$ (and therefore $D_{\hat{x}} = \frac{g_s}{\ell_s} D^1$). We also have to rescale time on the world-sheet $t = \frac{\ell_s^2}{R} \tau$. Moreover, we shall deal with dimensionless background target fields such that they will be measured in string units, *i.e.*, rescale (X, θ) to $(\ell_s X, \ell_s \theta)$. All this done, we are left with the rescaled action,

¹Throughout, derivatives with no explicit subscript shall refer to the cylinder world-sheet index σ , *i.e.*, $D \equiv D_\sigma$ and $\partial \equiv \partial_\sigma$.

$$\begin{aligned}
S' = & \frac{1}{2\pi} \int d\tau d\sigma \, \text{Tr} \left(\frac{1}{2} \dot{X}_i \dot{X}_i + 2\pi^2 g_s^2 \dot{A}^2 + \frac{1}{8\pi^2 g_s^2} \sum_{i < j} [X^i, X^j]^2 \right. \\
& \left. - \frac{1}{2} (DX^i)^2 + \frac{i}{4\pi R_9} \theta^T \dot{\theta} - \frac{1}{8\pi^2 g_s R_9} \theta^T \gamma_i [X^i, \theta] - \frac{i}{4\pi R_9} \theta^T \gamma_9 D\theta \right).
\end{aligned}$$

In order to cast this action into a more familiar looking one, we simply have to perform one further rescaling of the background fermions, $\theta \rightarrow \sqrt{4\pi R_9} \theta$, and change the notation for the string coupling constant as $g_s \rightarrow \frac{g_s}{2\pi}$. Then the 9–11 flip is concluded and we have obtained the DVV reduction of the Matrix theory action,

$$\begin{aligned}
S = & \frac{1}{2\pi} \int d\tau d\sigma \, \text{Tr} \left(\frac{1}{2} ((\dot{X}_i)^2 - (DX^i)^2) + \frac{1}{2g_s^2} \sum_{i < j} [X^i, X^j]^2 + \frac{1}{2} g_s^2 \dot{A}^2 \right. \\
& \left. + i(\theta^T \dot{\theta} - \theta^T \gamma_9 D\theta) - \frac{1}{g_s} \theta^T \gamma_i [X^i, \theta] \right). \tag{17}
\end{aligned}$$

This action is second quantized in the sense that it describes multiple interacting strings. One can further consider the special case of free strings, recovered in the infra-red limit with $g_s = 0$. In this limit the above two dimensional gauge theory becomes strongly coupled – as the Yang–Mills gauge coupling is related to the string coupling as $g_{YM} \sim \frac{1}{g_s}$ – and a non-trivial conformal field theory will describe the IR fixed point [19]. One can observe that in this limit, $g_s \rightarrow 0$, the world-sheet gauge field drops out, and moreover all matrices are diagonalized, *i.e.*, they will commute,

$$[X^i, X^j] = 0, \quad [X^i, \theta] = 0. \tag{18}$$

In this conformal field theory limit the previous Matrix string action reduces to,

$$S = \frac{1}{2\pi} \int d\tau d\sigma \left(\frac{1}{2} (\partial_\mu X^i)^2 + i\theta^T \rho^\mu \partial_\mu \theta \right), \tag{19}$$

where $\{\mu\}$ are world-sheet indices. This action can be exactly mapped to the light-cone Green-Schwarz action for the Type II superstring [24, 25, 26, 27, 28].

We have gone through a lengthy review of the DVV reduction, in order to set pace and notation for the section that follows. In there, we shall follow the same procedure applied to the full set of multipole moments of the 11-dimensional supercurrents for the stress tensor T^{IJ} , membrane current J^{IJK} and fivebrane current M^{IJKLMN} . These “DVV reduced” tensors will be the basis for the matrix string theory action in a weakly curved background.

4 Reduction via the 9–11 Flip

In order to write down the matrix string theory action in weak background fields, one needs to know the DVV reduction of the Matrix theory stress tensor, membrane current and 5-brane current. This should be clear from the fact that the matrix string theory action is

obtained via a DVV reduction of the Matrix theory action (as explained in the previous section), and the fact that in weak background fields the Matrix theory action is constructed precisely with the use of these tensors and currents (as explained in section 2, in particular in expression (3)). We will begin in here by applying the DVV reduction using the 9 – 11 flip, just as described previously. Later, we shall look at the sequence of dualities, and compare both procedures.

Let us start by specifying the conventions for the following of this section. Time derivatives are taken with respect to Minkowski time t . All expressions have been written in a gauge with $A_0 = 0$. Gauge invariance can be restored by replacing \dot{X} with $D_t X$. Indices i, j, \dots , run from 1 through 9, while indices a, b, \dots , run from 0 through 9. In these expressions we use the definitions $F_{0i} = \dot{X}^i$, $F_{ij} = i[X^i, X^j]$. A Matrix form for the transverse 5-brane current components M^{+ijklm} , M^{ijklmn} is as yet unknown. There are also fermionic components of the supercurrent which couple to background fermion fields in the supergravity theory. We will not discuss these couplings in this paper, but the Matrix theory form of the currents is determined in [14]. Moreover, there is also a 6-brane current appearing in Matrix theory related to nontrivial 11-dimensional background metrics.

4.1 Matrix Theory Tensors

We shall briefly describe the DVV reduction of the first component of the stress tensor, referring the specifics to the full description in the previous section. Then, we shall simply present the results for the other components in a schematic form (in the Appendix).

The zeroth moment of the T^{++} component of the Matrix stress tensor is given by,

$$T^{++} = \frac{1}{R} \mathbf{STr} (\mathbb{1}), \quad (20)$$

where \mathbf{STr} indicates a trace which is symmetrized over all orderings of terms of the forms F_{ab} , θ and $[X^i, \theta]$. We shall denote by the same name, T^{++} , the time integrated component which appears in the curved Matrix theory action. It is to this integrated term that we will apply the DVV reduction. In this term there is no need to introduce explicit dimensionfull parameters – there are no background fields – but we shall do it automatically in all the following terms, just as we did for the Matrix theory action in the previous section. As the theory is further compactified along the 9th direction, we T -dualize to obtain, after the 9 – 11 flip,

$$T^{++} = \int dt \frac{1}{R} \frac{g_s}{2\pi\ell_s} \int_0^{2\pi\frac{\ell_s}{g_s}} d\hat{x} \mathbf{STr} (\mathbb{1}). \quad (21)$$

Rescaling of world-sheet coordinates, background fields, and coupling constants (most of them trivial for this component), we are left with the final result,

$$T^{++} = \frac{1}{2\pi} \left(\frac{\ell_s}{R} \right)^2 \int d\sigma d\tau \mathbf{STr} (\mathbb{1}). \quad (22)$$

Moreover, we shall later be interested in the conformal field theory limit of these tensors. So, we further observe that the free string limit can be easily taken as,

$$\lim_{g_s \rightarrow 0} T^{++} = T^{++}. \quad (23)$$

We can proceed along the same line for the following components. The zeroth moment of the T^{+i} component of the Matrix stress tensor is given by,

$$T^{+i} = \frac{1}{R} \mathbf{STr} \left(\dot{X}_i \right). \quad (24)$$

Under T -duality for the 9 – 11 flip, one obtains for $i \neq 9$,

$$T^{+i} = \int dt \frac{1}{R} \frac{g_s}{2\pi \ell_s} \int_0^{2\pi \frac{\ell_s}{g_s}} d\hat{x} \mathbf{STr} \left(\dot{X}_i \right). \quad (25)$$

After the needed rescalings of world-sheet coordinates, background fields, and coupling constants, we are left with the final result,

$$T^{+i} = \frac{1}{2\pi} \left(\frac{\ell_s}{R} \right) \int d\sigma d\tau \mathbf{STr} \left(\dot{X}_i \right). \quad (26)$$

As to the free string limit, it can be taken as,

$$\lim_{g_s \rightarrow 0} T^{+i} = T^{+i}. \quad (27)$$

Under T -duality for the 9 – 11 flip, one obtains for $i = 9$,

$$T^{+9} = \int dt \frac{1}{R} \frac{g_s}{\ell_s} \int_0^{2\pi \frac{\ell_s}{g_s}} d\hat{x} \mathbf{STr} \left(\ell_s^2 \dot{A} \right). \quad (28)$$

After the needed rescalings of world-sheet coordinates, background fields, and coupling constants, we are left with the final result,

$$T^{+9} = \frac{1}{2\pi} \left(\frac{\ell_s}{R} \right) \int d\sigma d\tau \mathbf{STr} \left(g_s \dot{A} \right). \quad (29)$$

As to the free string limit, it can be taken as,

$$\lim_{g_s \rightarrow 0} T^{+9} = 0. \quad (30)$$

The procedure is always the same, for all the components. It should be clear to the reader how to obtain all the results, which are presented schematically in the Appendix. A few comments can be made, about the structures we have derived. First, as was trivially expected, the string coupling appears as expected, *i.e.*, a factor of g_s for each factor of \dot{A} , and factor of $\frac{1}{g_s}$ for each factor of $[X, X]$, or for each factor of $[X, \theta]$. Moreover, every tensor (and the action in section 3, also) has an overall normalization factor of $\frac{1}{2\pi}$. Second, and more importantly, we observe that if one counts operator insertions of background coordinates

into the currents as \dot{X} , \dot{A} , θ , $[X, X]$ and $[X, \theta]$ each counting as one operator insertion, then the order of the currents depends on the number of insertions as follows. For zero insertions, it is order $\mathcal{O} = (\frac{\ell_s}{R})^2$; for one insertion, it is order $\mathcal{O} = (\frac{\ell_s}{R})$; for two insertions, it is order $\mathcal{O} = 1$; for three insertions, it is order $\mathcal{O} = (\frac{R}{\ell_s})$; for four insertions, it is order $\mathcal{O} = (\frac{R}{\ell_s})^2$; and so on, for n insertions it is order $\mathcal{O} = (\frac{R}{\ell_s})^{n-2}$.

4.2 Matrix String Theory Tensors

We have thus performed the analysis of the Matrix theory expressions for the stress tensor, the membrane current and the 5-brane current. As previously explained in section 2, one can obtain the matrix string theory action in terms of other tensors: the sources I_p of Dp -brane currents for $p = 2n$, the sources I_s and I_5 associated with fundamental string and $NS5$ -brane currents respectively, and also the sources I_h and I_ϕ of background metric and background dilaton fields. These currents I can moreover be expressed as linear combinations of the Matrix theory expressions for T , J and M . In previous work, the results for the lowest dimension operators appearing in the monopole (integrated) $D0$ -brane currents were obtained [14, 34, 35].

Now, because of the 9 – 11 flip, we are dealing with M -theory on spacelike R_9 instead of M -theory on spacelike R_{11} as before. This means that the I tensors are not necessarily related to the T , J and M tensors in the same way as in the case of the $D0$ -brane action that was described briefly in section 2. We begin by addressing such a question, in order to derive the correct expressions for the I linear tensor couplings. The original M -theory where the $D0$ -brane couplings were derived was spacelike compactified along R_{11} , so that in light-cone coordinates we would be dealing with $X^\pm \sim X^0 \pm X^{11}$ and a further compact coordinate X^9 . With the 9 – 11 flip we are now led to an \hat{M} -theory compactified along R_9 , and where in light-cone gauge the coordinates are now $\hat{X}^\pm \sim \hat{X}^0 \pm \hat{X}^9$ and the compact coordinate \hat{X}^{11} . Clearly we have two frames, the “11” frame in the original M -theory, and the “9” frame in the flipped \hat{M} -theory. The relations we presented briefly in section 2 concerning the relation between the I tensors and the Matrix theory tensors T , J and M , are still valid in the flipped “9” frame, but now relating the I tensors to the Matrix tensors in this frame, *i.e.*, \hat{T} , \hat{J} and \hat{M} . If we then relate these “9” frame Matrix tensors back to the “11” frame Matrix tensors, we will be able to express the matrix string theory couplings I in terms of the just derived DVV reduced Matrix tensors T , J and M . So, all one needs to do is a simple change of coordinates.

To begin with a simple example, let us look at the I_s^{ij} component of the matrix fundamental string current, which is given by:

$$I_s^{ij} \sim \hat{J}^{+ij} - \hat{J}^{-ij}. \quad (31)$$

This expression holds in the “9” frame. Relating the \hat{J} tensor to the J tensor in the “11” frame, one obtains,

$$I_s^{ij} \sim \hat{J}^{+ij} - \hat{J}^{-ij} \sim J^{9ij}. \quad (32)$$

On the other hand, at the level of background fields, one knows how to relate the NS 2-form $B_{\mu\nu}$ to the M -theory 3-form A_{IJK} , via $B_{ij} \sim A_{9ij}$, where 9 is the spacelike compact direction involved in the Sen–Seiberg limit. The coupling we have just derived above is then precisely what one would expect.

The procedure is always the same, and it should be straightforward for the reader to reproduce the results, which we now present schematically. Observe that as we change from the “9” frame to the “11” frame, there is a mixing of different orders in the currents, *i.e.*, there will be tensors in different powers of $\mathcal{O}\left(\left(\frac{R}{\ell_s}\right)^{n-2}\right)$. We will neglect some of these tensors in the following expressions, and only keep the orders of interest to us. The components of the matrix string current associated with the background metric field are:

$$\begin{aligned} I_h^{00} &= \hat{T}^{++} + \hat{T}^{+-} + (I_h^{00})_8 + \mathcal{O}(\hat{X}^{12}) = T^{++} + T^{+-} + \dots \\ &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\left(\frac{\ell_s}{R} \right)^2 \mathbb{1} + \frac{1}{2} \dot{X}_i^2 + \frac{1}{2} (DX^i)^2 + \frac{1}{2} g_s^2 \dot{A}^2 - \frac{1}{2g_s^2} \sum_{i<j} [X^i, X^j]^2 \right. \\ &\quad \left. + \frac{1}{g_s} \theta \gamma_i [X^i, \theta] + i\theta \gamma^9 D\theta \right) + \dots, \\ I_h^{0i} &= \hat{T}^{+i} + \hat{T}^{-i} + \mathcal{O}(\hat{X}^{10}) = T^{+i} + T^{-i} + \dots \\ &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left\{ \left(\frac{\ell_s}{R} \right) \dot{X}_i + \left(\frac{R}{\ell_s} \right) \left[\frac{1}{2} \dot{X}_i (\dot{X}_j)^2 + \frac{1}{2} g_s^2 \dot{X}_i \dot{A}^2 - \frac{1}{2g_s^2} \dot{X}_i \sum_{j<k} [X^j, X^k]^2 \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \dot{X}_i (DX^j)^2 - \frac{1}{g_s^2} [X^i, X^j] [X^j, X^k] \dot{X}_k - DX^i DX^k \dot{X}_k + i[X^i, X^j] DX^j \dot{A} \right. \right. \\ &\quad \left. \left. - \frac{1}{2g_s} \theta_\alpha \dot{X}_k [X_j, \theta_\beta] \{ \gamma^k \delta_{ij} + \gamma^i \delta_{jk} - 2\gamma^j \delta_{ki} \}_{\alpha\beta} - \frac{1}{2} \dot{A} \theta \gamma^9 [X_i, \theta] + i\dot{X}_i \theta \gamma^9 D\theta \right. \right. \\ &\quad \left. \left. - \frac{i}{2} g_s \dot{A} \theta \gamma^i D\theta - \frac{i}{4g_s^2} \theta_\alpha [X^k, X^j] [X^l, \theta_\beta] \{ \gamma^{[ikjl]} + 2\gamma^{[jl]} \delta_{ki} + 4\delta_{ki} \delta_{jl} \}_{\alpha\beta} \right. \right. \\ &\quad \left. \left. - \frac{1}{2g_s} \theta_\alpha DX^k [X^j, \theta_\beta] \{ \gamma^{[ik9j]} + \gamma^{[9j]} \delta_{ki} \}_{\alpha\beta} + \frac{1}{4g_s} \theta_\alpha [X^k, X^j] D\theta_\beta \{ \gamma^{[ikj9]} + 2\gamma^{[j9]} \delta_{ki} \}_{\alpha\beta} \right. \right. \\ &\quad \left. \left. - iDX^i \theta D\theta + \dots \right\} + \dots, \\ I_h^{ij} &= \hat{T}^{ij} + (I_h^{ij})_8 + \mathcal{O}(\hat{X}^{12}) = T^{ij} + \dots \\ &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\dot{X}_i \dot{X}_j - DX^i DX^j - \frac{1}{g_s^2} [X^i, X^k] [X^k, X^j] \right. \\ &\quad \left. - \frac{1}{2g_s} \theta \gamma^i [X_j, \theta] - \frac{1}{2g_s} \theta \gamma^j [X_i, \theta] \right) + \dots, \end{aligned} \quad (33)$$

where $(I_h^{00})_8 = \frac{3}{2} \hat{T}^{--} + \dots$ and $(I_h^{ij})_8 = 2\hat{T}^{--} + \dots$, and we know the matrix string form of \hat{T}^{--} from (80). The conformal field theory limit of these tensors is simply:

$$\begin{aligned}
\lim_{g_s \rightarrow 0} I_h^{00} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\left(\frac{\ell_s}{R} \right)^2 \mathbb{1} + \frac{1}{2} \dot{X}_i^2 + \frac{1}{2} (\partial X^i)^2 + i\theta \gamma^9 \partial \theta \right) + \dots, \\
\lim_{g_s \rightarrow 0} I_h^{0i} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\left(\frac{\ell_s}{R} \right) \dot{X}_i + \left(\frac{R}{\ell_s} \right) \left\{ \frac{1}{2} \dot{X}_i (\dot{X}_j)^2 + \frac{1}{2} \dot{X}_i (\partial X^j)^2 - \partial X^i \partial X^k \dot{X}_k \right. \right. \\
&\quad \left. \left. + i \dot{X}_i \theta \gamma^9 \partial \theta - i \partial X^i \theta \partial \theta \right\} \right) + \dots, \\
\lim_{g_s \rightarrow 0} I_h^{ij} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\dot{X}_i \dot{X}_j - \partial X^i \partial X^j \right) + \dots
\end{aligned} \tag{34}$$

The I_ϕ matrix string current associated to the dilaton is given by,

$$\begin{aligned}
I_\phi &= \hat{T}^{++} - \frac{1}{3}(\hat{T}^{+-} + \hat{T}^{ii}) + (I_\phi)_8 + \mathcal{O}(\hat{X}^{12}) = T^{++} + \frac{1}{3}(T^{+-} + T^{ii}) + \dots \\
&= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\left(\frac{\ell_s}{R} \right)^2 \mathbb{1} + \frac{1}{2} \dot{X}_i^2 - \frac{1}{2} (DX^i)^2 + \frac{1}{2} g_s^2 \dot{A}^2 + \frac{1}{2g_s^2} \sum_{i < j} [X^i, X^j]^2 \right) \\
&\quad + \dots,
\end{aligned} \tag{35}$$

where $(I_\phi)_8 = -\frac{1}{2}\hat{T}^{--} + \dots$ and we know the matrix string form of \hat{T}^{--} from (80). This current has the following conformal field theory limit,

$$\lim_{g_s \rightarrow 0} I_\phi = \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\left(\frac{\ell_s}{R} \right)^2 \mathbb{1} + \frac{1}{2} \dot{X}_i^2 - \frac{1}{2} (\partial X^i)^2 \right) + \dots \tag{36}$$

The components of the matrix fundamental string current are:

$$\begin{aligned}
I_s^{0i} &= 3\hat{J}^{+-i} + \mathcal{O}(\hat{X}^8) = 3J^{+i9} + 3J^{-i9} + \dots \\
&= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{1}{2} \left(\frac{\ell_s}{R} \right) DX^i + \left(\frac{R}{\ell_s} \right) \left\{ \frac{1}{2} \dot{X}^i \dot{X}^k DX^k - \frac{i}{2} \dot{A} \dot{X}^k [X^k, X^i] + \frac{g_s^2}{4} \dot{A}^2 DX^i \right. \right. \\
&\quad \left. \left. - \frac{1}{4} (\dot{X}^k)^2 DX^i - \frac{1}{4g_s^2} DX^i \sum_{k < l} [X^k, X^l]^2 - \frac{1}{4} DX^i (DX^k)^2 \right. \right. \\
&\quad \left. \left. - \frac{1}{2g_s^2} [X^i, X^k] [X^k, X^l] DX^l + \frac{1}{4g_s} \theta_\alpha \dot{X}^k [X^m, \theta_\beta] \{ \gamma^{[ki9m]} + \gamma^{[9m]} \delta_{ki} \} \}_{\alpha\beta} \right. \right. \\
&\quad \left. \left. - \frac{1}{4} \dot{A} \theta_\alpha [X^m, \theta_\beta] \{ \gamma^{[im]} + 2\delta_{im} \} \}_{\alpha\beta} + \frac{i}{2} \dot{X}^i \theta D\theta - \frac{i}{4} \dot{A} \theta \gamma^{[i9]} D\theta \right. \right. \\
&\quad \left. \left. + \frac{3i}{4g_s^2} \theta_\alpha [X^k, X^l] [X^m, \theta_\beta] \{ \gamma^{[9kl]} \delta_{mi} + 2\gamma^{[li9]} \delta_{km} + 2\gamma^9 \delta_{il} \delta_{km} \} \}_{\alpha\beta} \right. \right. \\
&\quad \left. \left. - \frac{3}{2g_s} DX^k \theta \gamma^k [X^i, \theta] - \frac{3}{2g_s} DX^k \theta \gamma^i [X^k, \theta] \right. \right. \\
&\quad \left. \left. + \frac{3i}{4g_s} \theta_\alpha [X^k, X^l] D\theta_\beta \{ \gamma^{[ikl]} + 2\gamma^l \delta_{ik} \} \}_{\alpha\beta} - i DX^i \theta \gamma^9 D\theta + \dots \right) \right) + \dots,
\end{aligned}$$

$$\begin{aligned}
I_s^{ij} &= 3\hat{J}^{+ij} - 3\hat{J}^{-ij} + \mathcal{O}(\hat{X}^{10}) = 3J^{9ij} + \dots \\
&= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{1}{2} \dot{X}^i D X^j + \frac{1}{2} \dot{X}^j D X^i - \frac{i}{2} \dot{A} [X^i, X^j] + \frac{1}{4g_s} \theta \gamma^{[ij9l]} [X_l, \theta] \right). \quad (37)
\end{aligned}$$

The conformal field theory limit of these tensors is:

$$\begin{aligned}
\lim_{g_s \rightarrow 0} I_s^{0i} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{1}{2} \left(\frac{\ell_s}{R} \right) \partial X^i + \left(\frac{R}{\ell_s} \right) \left\{ \frac{1}{2} \dot{X}^i \dot{X}^k \partial X^k \right. \right. \\
&\quad \left. \left. - \frac{1}{4} (\dot{X}^k)^2 \partial X^i - \frac{1}{4} \partial X^i (\partial X^k)^2 + \frac{i}{2} \dot{X}^i \theta \partial \theta - i \partial X^i \theta \gamma^9 \partial \theta \right\} \right) + \dots, \\
\lim_{g_s \rightarrow 0} I_s^{ij} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{1}{2} \dot{X}^i \partial X^j + \frac{1}{2} \dot{X}^j \partial X^i \right) + \dots \quad (38)
\end{aligned}$$

Let us now move to the R-R fields. The components of the matrix string $D0$ -brane current are:

$$\begin{aligned}
I_0^0 &= \hat{T}^{++} = T^{+9} + T^{-9} + \dots \\
&= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\left(\frac{\ell_s}{R} \right) g_s \dot{A} + \left(\frac{R}{\ell_s} \right) \left\{ \frac{1}{2} g_s \dot{A} (\dot{X}^i)^2 + \frac{1}{2} g_s^3 \dot{A}^3 - \frac{1}{2g_s} \dot{A} \sum_{i < j} [X^i, X^j]^2 \right. \right. \\
&\quad \left. \left. - \frac{i}{g_s} D X^i [X^i, X^j] \dot{X}^j - \frac{1}{2} g_s (D X^i)^2 \dot{A} - \frac{1}{2g_s} \dot{X}^i \theta \gamma^9 [X^i, \theta] + \dot{A} \theta \gamma^i [X^i, \theta] \right. \right. \\
&\quad \left. \left. - \frac{i}{2} \dot{X}^i \theta \gamma^i D \theta - \frac{i}{4g_s^2} [X^i, X^j] \theta \gamma^{[9ijk]} [X^k, \theta] + \frac{1}{2g_s} \theta_\alpha D X^i [X^j, \theta_\beta] \{ \gamma^{[ij]} + 2\delta_{ij} \}_{\alpha\beta} \right. \right. \\
&\quad \left. \left. + \frac{i}{2} D X^i \theta \gamma^{[i9]} D \theta + \dots \right\} \right) + \dots, \\
I_0^i &= \hat{T}^{+i} = T^{9i} + \dots \\
&= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(g_s \dot{X}_i \dot{A} + \frac{i}{g_s} [X^i, X^k] D X^k - \frac{i}{2} \theta \gamma^i D \theta - \frac{1}{2g_s} \theta \gamma^9 [X_i, \theta] \right) + \dots \quad (39)
\end{aligned}$$

With the conformal field theory limit:

$$\begin{aligned}
\lim_{g_s \rightarrow 0} I_0^0 &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(-\frac{i}{2} \dot{X}^i \theta \gamma^i \partial \theta + \frac{i}{2} \partial X^i \theta \gamma^{[i9]} \partial \theta \right) + \dots, \\
\lim_{g_s \rightarrow 0} I_0^i &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{i}{2} \theta \gamma^i \partial \theta \right) + \dots \quad (40)
\end{aligned}$$

Observe that these expressions for the $D0$ -brane current are exact in the hatted frame, unlike all the other expressions for the matrix string theory currents, which are given up to higher order terms in the coordinate fields, $\mathcal{O}(\hat{X}^n)$.

The components of the matrix string theory $D2$ -brane current are:

$$\begin{aligned}
I_2^{0ij} &= \hat{J}^{+ij} + \mathcal{O}(\hat{X}^{10}) = J^{+ij} + J^{-ij} + \dots \\
&= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\left(\frac{\ell_s}{R}\right) \frac{i}{6g_s} [X^i, X^j] + \left(\frac{R}{\ell_s}\right) \left\{ \frac{i}{6g_s} \dot{X}^i \dot{X}^k [X^k, X^j] \right. \right. \\
&\quad - \frac{i}{6g_s} \dot{X}^j \dot{X}^k [X^k, X^i] - \frac{1}{6} g_s \dot{A} \dot{X}^i D X^j + \frac{1}{6} g_s \dot{A} \dot{X}^j D X^i \\
&\quad - \frac{i}{12g_s} (\dot{X}^k)^2 [X^i, X^j] - \frac{i}{12} g_s \dot{A}^2 [X^i, X^j] - \frac{i}{12g_s^3} [X^i, X^j] \sum_{k < l} [X^k, X^l]^2 \\
&\quad + \frac{i}{12g_s} [X^i, X^j] (D X^k)^2 - \frac{i}{6g_s^3} [X^i, X^k] [X^k, X^l] [X^l, X^j] - \frac{i}{6g_s} D X^i D X^k [X^k, X^j] \\
&\quad + \frac{i}{6g_s} D X^j D X^k [X^k, X^i] + \frac{1}{12g_s} \theta_\alpha \dot{X}^k [X^m, \theta_\beta] \{ \gamma^{[kijm]} + \gamma^{[jmi]} \delta_{ki} - \gamma^{[im]} \delta_{kj} \\
&\quad \quad \quad + 2\delta_{jm} \delta_{ki} - 2\delta_{im} \delta_{kj} \}_{\alpha\beta} \\
&\quad + \frac{1}{12} \dot{A} \theta \gamma^{[9ijm]} [X^m, \theta] + \frac{i}{12} \theta_\alpha \dot{X}^k D \theta_\beta \{ \gamma^{[kij9]} + \gamma^{[j9]} \delta_{ki} - \gamma^{[i9]} \delta_{kj} \}_{\alpha\beta} \\
&\quad + \frac{i}{4g_s^2} \theta_\alpha [X^k, X^l] [X^m, \theta_\beta] \{ \gamma^{[jkl]} \delta_{mi} - \gamma^{[ikl]} \delta_{mj} + 2\gamma^{[lij]} \delta_{km} + 2\gamma^l \delta_{jk} \delta_{im} \\
&\quad \quad \quad - 2\gamma^l \delta_{ik} \delta_{jm} + 2\gamma^j \delta_{il} \delta_{km} - 2\gamma^i \delta_{jl} \delta_{km} \}_{\alpha\beta} \\
&\quad + \frac{1}{2g_s} \theta_\alpha D X^k [X^m, \theta_\beta] \{ \gamma^{[jk9]} \delta_{mi} - \gamma^{[ik9]} \delta_{mj} + \gamma^{[9ij]} \delta_{km} + \gamma^9 \delta_{jk} \delta_{im} - \gamma^9 \delta_{ik} \delta_{jm} \}_{\alpha\beta} \\
&\quad \left. - \frac{i}{2} \theta_\alpha D X^l D \theta_\beta \{ \gamma^{[lij]} + \gamma^j \delta_{il} - \gamma^i \delta_{jl} \}_{\alpha\beta} + \dots \right) + \dots, \\
I_2^{ijk} &= \hat{J}^{ijk} + \mathcal{O}(\hat{X}^8) = J^{ijk} + \dots \\
&= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{i}{6g_s} \dot{X}^i [X^j, X^k] - \frac{i}{6g_s} \dot{X}^j [X^k, X^i] - \frac{i}{6g_s} \dot{X}^k [X^i, X^j] \right. \\
&\quad \left. + \frac{1}{12g_s} \theta \gamma^{[ijk]} [X_l, \theta] + \frac{i}{12} \theta \gamma^{[ijk9]} D \theta \right) + \dots
\end{aligned} \tag{41}$$

The conformal field theory limit for the membrane current is:

$$\begin{aligned}
\lim_{g_s \rightarrow 0} I_2^{0ij} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(\frac{i}{12} \theta_\alpha \dot{X}^k \partial \theta_\beta \{ \gamma^{[kij9]} + \gamma^{[j9]} \delta_{ki} - \gamma^{[i9]} \delta_{kj} \}_{\alpha\beta} \right. \\
&\quad \left. - \frac{i}{2} \theta_\alpha \partial X^l \partial \theta_\beta \{ \gamma^{[lij]} + \gamma^j \delta_{il} - \gamma^i \delta_{jl} \}_{\alpha\beta} \right) + \dots, \\
\lim_{g_s \rightarrow 0} I_2^{ijk} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\frac{i}{12} \theta \gamma^{[ijk9]} \partial \theta \right) + \dots
\end{aligned} \tag{42}$$

Moving towards the $D4$ -brane current, the components are given by:

$$\begin{aligned}
I_4^{0ijkl} &= 6\hat{M}^{+-ijkl} + \mathcal{O}(\hat{X}^8) = 6M^{-i9jkl} + \dots \\
&= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(-\frac{30i}{g_s} \dot{X}^i [X^j, X^k] D X^l - \frac{15}{2g_s} \dot{A} [X^i, X^j] [X^l, X^k] \right)
\end{aligned}$$

$$\begin{aligned}
& +5i\theta\dot{X}^{[i}\gamma^{jkl]}D\theta - 5\dot{A}\theta\gamma^{[ijl}[X^k],\theta] - \frac{15}{g_s}\theta\dot{X}^{[i}\gamma^{9]kl}[X^j],\theta] \\
& + \frac{15i}{2g_s^2}\theta[X^i,X^j]\gamma^{kl9}\gamma^n[X^n,\theta] - \frac{15}{2g_s}\theta[X^i,X^j]\gamma^{kl9}\gamma^9D\theta \\
& + \frac{5}{g_s}\theta DX^{[i}\gamma^{jlk]}\gamma^n[X^n,\theta] + 5i\theta DX^{[i}\gamma^{jlk]}\gamma^9D\theta) + \dots, \\
I_4^{ijklm} &= -6\hat{M}^{-ijklm} + \mathcal{O}(\hat{X}^{10}) = -6M^{-ijklm} + \dots \\
&= \frac{1}{2\pi}\left(\frac{R}{\ell_s}\right)\int d\sigma d\tau \mathbf{STr}\left(\frac{15}{2g_s^2}\dot{X}^{[i}[X^j,X^k][X^l,X^m] + \frac{5}{g_s}\theta\dot{X}^{[i}\gamma^{jkl}[X^m],\theta] \right. \\
& \left. + \frac{5i}{2g_s^2}\theta[X^i,X^j]\gamma^{klm}\gamma^n[X^n,\theta] - \frac{5}{2g_s}\theta[X^i,X^j]\gamma^{klm}\gamma^9D\theta) + \dots \tag{43}
\end{aligned}$$

The conformal field theory limit for the 4-brane current is:

$$\begin{aligned}
\lim_{g_s \rightarrow 0} I_4^{0ijkl} &= \frac{1}{2\pi}\left(\frac{R}{\ell_s}\right)\int d\sigma d\tau \mathbf{STr} 5i\left(\theta\dot{X}^{[i}\gamma^{jkl]}\partial\theta + \theta\partial X^{[i}\gamma^{jlk]}\gamma^9\partial\theta\right) + \dots, \\
\lim_{g_s \rightarrow 0} I_4^{ijklm} &= \mathcal{O}(X^{10}) + \dots \tag{44}
\end{aligned}$$

Next we analyze the $D6$ -brane current in matrix string theory. The components of this current are given by:

$$\begin{aligned}
I_6^{0ijklmn} &= \hat{S}^{+ijklmn} + \mathcal{O}(\hat{X}^{10}) = S^{+ijklmn} + \dots \\
&= \frac{1}{2\pi}\left(\frac{R}{\ell_s}\right)\int d\sigma d\tau \mathbf{STr}\left(-\frac{i}{g_s^3}[X^i,X^j][X^k,X^l][X^m,X^n]\right) + \dots, \\
I_6^{ijklmnp} &= \hat{S}^{ijklmnp} + \mathcal{O}(\hat{X}^{12}) = S^{ijklmnp} + \dots \\
&= \frac{1}{2\pi}\left(\frac{R}{\ell_s}\right)^2\int d\sigma d\tau \mathbf{STr}\left(-\frac{7i}{g_s^3}[X^i,X^j][X^k,X^l][X^m,X^n]\dot{X}^p + \mathcal{O}(\theta^2,\theta^4)\right) \\
&+ \dots \tag{45}
\end{aligned}$$

As to the conformal field theory limit for the 6-brane current, it is:

$$\begin{aligned}
\lim_{g_s \rightarrow 0} I_6^{0ijklmn} &= \mathcal{O}(X^{10}) + \dots, \\
\lim_{g_s \rightarrow 0} I_6^{ijklmnp} &= \mathcal{O}(X^{12}) + \dots \tag{46}
\end{aligned}$$

A Matrix theory form for the transverse $M5$ -brane current components M^{+ijklm} and M^{ijklmn} is not known (though it is believed that these operators identically vanish in this light-front gauge). Therefore, we cannot determine the $NS5$ -brane current components I_5^{ijklmn} and I_5^{0ijklm} (though most likely these operators will vanish in the Type IIA description as well, at least to the lowest order we are considering here). For further discussions on these points, see [34].

With these matrix string currents, the sigma model action for matrix string theory in weakly curved backgrounds is then simply written as:

$$\begin{aligned} \mathcal{S} = & \frac{1}{2\pi} \int d\sigma d\tau \left(\frac{1}{2} g_{\mu\nu}^{IIA}(X) I_g^{\mu\nu} + \phi(X) I_\phi + B_{\mu\nu}(X) I_s^{\mu\nu} + \tilde{B}_{\mu\nu\lambda\rho\sigma\tau}(X) I_5^{\mu\nu\lambda\rho\sigma\tau} \right. \\ & \left. + C_\mu(X) I_0^\mu + \tilde{C}_{\mu\nu\lambda\rho\sigma\tau\xi}(X) I_6^{\mu\nu\lambda\rho\sigma\tau\xi} + C_{\mu\nu\lambda}(X) I_2^{\mu\nu\lambda} + \tilde{C}_{\mu\nu\lambda\rho\sigma}(X) I_4^{\mu\nu\lambda\rho\sigma} \right), \end{aligned} \quad (47)$$

where the notation is as follows. The metric is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, so that the first term includes naturally, to linear order in $h_{\mu\nu}$, the term relative to the matrix string action in flat space and the linear coupling term $h_{\mu\nu}(X) I_h^{\mu\nu}$ previously derived. Also, we have seen that the currents I that we derived were integrated currents, including an explicit world-sheet integration. In (47) this world-sheet integration, as well as the $\frac{1}{2\pi}$ factor, have been brought out of the expressions for the current, in order to stress that we have obtained a two dimensional matrix gauged sigma model field theory. Finally, recall from [14, 34, 35, 41] what is the prescription to include explicit spacetime dependence in all the NS-NS and R-R fields. We should take the following definition:

$$\phi(X) I_\phi \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (\partial_{k_1} \cdots \partial_{k_n} \phi)(0) I_\phi^{(k_1 \cdots k_n)}, \quad (48)$$

where $I_\phi^{(k_1 \cdots k_n)}$ are the higher moments of the matrix string current for the dilaton. Similarly for all the other fields. Observe that in this work we have just analyzed zeroth moments of the Matrix and matrix string currents. Moreover, we shall assume for the remainder of the paper that the background fields satisfy the source-free equations of motion of Type IIA supergravity. In this case, the dual fields \tilde{C}^{D6} , \tilde{B}^{NS5} and \tilde{C}^{D4} are well defined $(p+1)$ -form fields given at linear order by,

$$d\tilde{C}^{D6} = \star dC^{D0}, \quad d\tilde{B}^{NS5} = \star dB, \quad d\tilde{C}^{D4} = \star dC^{D2}. \quad (49)$$

In one sentence, the tensors we have just derived allow us to build a matrix sigma model for the IIA string. Recall from the standard sigma model approach that the background fields are an infinite number of couplings from the point of view of the world-sheet quantum field theory. If the target space has characteristic curvature R , then derivatives of the metric will be of order $\frac{1}{R}$, and so the effective dimensionless coupling in the theory will be $\frac{\sqrt{\alpha'}}{R} = \frac{\ell_s}{R}$, quite similar to what we have obtained (even though previously R was simply the radius of the compact dimension). For $R \gg \ell_s$ the effective coupling is small and perturbation theory on the world-sheet is useful. In this regime, the string is effectively point-like, and one can also use the low energy effective field theory to deal with the problem. To these results there will naturally be stringy corrections. They can be obtained from the multi-loop corrections to the world-sheet beta functions, as a power series in our effective coupling $\frac{\ell_s}{R}$.

One expects that a similar story should take place in the case of the matrix sigma model we have derived, that describes multiple string interactions in non-trivial background fields.

Indeed, it would be quite interesting to derive matrix beta functions, and therefore matrix equations of motion for the background fields. While at the level of Matrix theory one could expect that a large N renormalization group analysis should be required [15], here at the level of matrix string theory it should only be a direct generalization of the one string sigma model field theory.

5 Reduction via the T - S - T Duality Sequence

If one recalls section 3, performing the DVV reduction via the 9–11 flip should be equivalent to a specific set of dualities, namely T -duality from IIA to IIB, S -duality of IIB, and then T -duality back to the IIA theory. We would now like to explicitly check such a procedure in the presence of non-trivial backgrounds. This would amount to a further check of the internal structure of string dualities. Basically, all one needs to do is DVV reduce according to the sequence of dualities as applied to the background fields and to the world-volume fields, and observe that one will obtain the same result as in the previous section. The transformation of the background fields under T and S dualities is well known. As to the transformation of the world-volume fields, it is well known for the case of T -duality as discussed for the general case in [22]. For S -duality, it is not known how the world-volume fields transform, as we are dealing with a nonabelian gauge theory. For the case of the $D3$ -brane, this has been discussed recently in [35]. In here, we shall be dealing with a $D1$ -brane. We will obtain the S -duality transformations for the world-volume fields of the 2-dimensional gauge theory by demanding consistency with the whole structure of matrix string theory. Also, we shall work this out explicitly only for a couple of terms, not for the whole series of components of the several matrix string supercurrents. Moreover, we will neglect all terms involving fermions throughout this section. From the previous section we already know how they appear in the tensor structures that compose matrix string theory, so that we can always take them from there when they are required in the following sections. However, for the purpose of checking the duality sequence it should be enough to look at the bosonic part of the action alone. Extending our results to all the components and including fermions should present no obvious obstacles.

We begin by following closely [35], in particular their discussion of implementing T and S dualities for the linear supergravity backgrounds, and somewhat the implementation of T -duality at the world-volume level. Then, due to the matching between the duality sequence and the 9 – 11 flip, we determine the S -duality transformation rules at the level of the 2-dimensional world-volume theory.

Let us focus on T -duality, and how it acts both on the supergravity background fields and on the fields that live on the D -brane. Recall that when we begin the sequence of dualities we are looking at the world-volume theory of $D0$ -branes. T -duality acting on arbitrary backgrounds independent of the compact directions is well known in string theory [42]. The standard T -duality rules can be linearized and these are the transformations we

shall be interested in, given that we are working in weak background fields. Using barred indices $\bar{\alpha}, \bar{\beta}, \dots$, for the compact directions in which a T -duality is performed and indices μ, ν, \dots , for the remaining $10-p$ spacetime dimensions (including 0), we can write the action of T -duality in the linear supergravity background fields as [35],

$$\begin{aligned}
h_{\mu\nu} &\longleftrightarrow h_{\mu\nu} \\
B_{\mu\nu} &\longleftrightarrow B_{\mu\nu} \\
h_{\mu\bar{\alpha}} &\longleftrightarrow -B_{\mu\bar{\alpha}} \\
h_{\bar{\alpha}\mu} &\longleftrightarrow B_{\bar{\alpha}\mu} \\
h_{\bar{\alpha}\bar{\beta}} &\longleftrightarrow -h_{\bar{\alpha}\bar{\beta}} \\
B_{\bar{\alpha}\bar{\beta}} &\longleftrightarrow -B_{\bar{\alpha}\bar{\beta}} \\
\phi &\longleftrightarrow \phi - \frac{1}{2} \sum_{\bar{\alpha}} h_{\bar{\alpha}\bar{\alpha}} \\
C_{\mu_1 \dots \mu_{q-k} \bar{\alpha}_1 \dots \bar{\alpha}_k}^{(q)} &\longleftrightarrow \frac{1}{(n-k)!} \epsilon^{\bar{\alpha}_1 \dots \bar{\alpha}_k} C_{\mu_1 \dots \mu_{q-k} \bar{\alpha}_{k+1} \dots \bar{\alpha}_n}^{(q-2k+n)}
\end{aligned} \tag{50}$$

where the (q) superscript indicates the q -form R-R field associated to a $D(q-1)$ -brane.

The implementation being clear at the background level, let us look at the world-volume level. The low energy effective field theory living on the world-volume of a Dp -brane in flat background space is the dimensional reduction of 10-dimensional SYM theory to the $p+1$ world-volume dimensions. One thing one can do [35] is to retain 10-dimensional notation for all the Dp -brane world-volume theories and reinterpret the resulting expressions appropriately for each case, *i.e.*, if we choose a, b, \dots , as world-volume indices and i, j, \dots , as indices transverse to the brane, then we would reinterpret expressions such as $F_{ai} \equiv D_a X^i$ and $F_{ij} \equiv i[X^i, X^j]$. We therefore see that the action of T -duality on expressions which have been written in terms of the 10-dimensional notation simply amounts to an adequate reinterpretation of such a notation. There is only one point one should take into account, namely we should be careful when considering transverse fields X^i associated with a compact direction. The precise way in which one should deal with such fields has been described in [22]. Briefly stated, transverse fields associated with a compact direction can be Fourier decomposed so that they are T -dual to the momentum modes of the dual gauge field that lives on the T -dual brane. The Matrix theory expressions for the moments of the 11-dimensional supergravity currents that we have used in the previous section can all be written easily in the 10-dimensional language, and this has been done in [35]. We refer to the appendix for those expressions.

Given the $D0$ -brane action in weak background fields, we can now write down the T -dual action for the Type IIB D -string. This has in fact been done for any Dp -brane [35], as briefly discussed in section 2. The D -string action is therefore,

$$\begin{aligned}
S_{NS-NS}^{D1} = & (\phi - \frac{1}{2}h_{\hat{a}\hat{a}})I_\phi + \frac{1}{2}h_{00}I_h^{00} + \frac{1}{2}h_{ij}I_h^{ij} - \frac{1}{2}h_{\hat{a}\hat{b}}I_h^{\hat{a}\hat{b}} + h_{0i}I_h^{0i} + 2h_{\hat{a}i}I_s^{\hat{a}i} - 2h_{0\hat{a}}I_s^{0\hat{a}} \\
& + B_{ij}I_s^{ij} - B_{\hat{a}\hat{b}}I_s^{\hat{a}\hat{b}} + 2B_{0i}I_s^{0i} + B_{\hat{a}i}I_h^{\hat{a}i} - B_{0\hat{a}}I_h^{0\hat{a}} \\
& + \text{Higher moment terms} + \text{Nonlinear terms},
\end{aligned} \tag{51}$$

$$\begin{aligned}
S_{R-R}^{D1} = & \int d^2\sigma \epsilon^{a_0 a_1} \sum_q \sum_{n=\text{Max}(0, q-2)}^{n=\text{Min}(q, 8)} \frac{(-1)^{\frac{n(n-1)}{2}} (2n-q+1)!!}{n!(q-n)!(n-q+2)!} \mathbf{STr} \{ C_{a_0 \dots a_{q-n-1} i_1 \dots i_n}^{(q)} \cdot \\
& \cdot F_{(a_{q-n} \dots a_1 i_1 \dots i_n)}^{n+\frac{2-q}{2}} \} + \text{Higher moment terms} + \text{Nonlinear terms},
\end{aligned} \tag{52}$$

where the indices in curved brackets are to be assigned pairwise to the corresponding product of F 's and then symmetrized over all orderings. Indices $0, \hat{a}, \dots$, live on the 2-dimensional world-volume, while indices i, j, \dots , are transverse to the D -string. The I currents in these expressions should not be confused with the I currents derived in the previous section. In here we started with the $D0$ -brane currents mentioned in section 2, which in turn can be determined in terms of the Matrix currents T , J and M . As we wrote the previous expressions, the 0-brane currents I are then to be reinterpreted as 10-dimensional expressions reduced to the 2-dimensional world-volume of the $D1$ -brane. As to the higher moment terms, the expressions will be just like the ones above, but with the appropriate inclusion of arbitrary derivatives of each background field.

It is therefore clear that in order to explicitly write the D -string action, it would be useful to start with the 10-dimensional expressions for the I currents. These can be obtained from the expressions in [35], and they are as follows. Observe that we write down the expressions in the previously explained 10-dimensional notation, and so when reducing to the 2-dimensional world-volume one should take into consideration the compact direction carefully, as was mentioned before.

The I_h^{00} component of the matrix string current for background metric field, can be written in 10-dimensional notation as (we dropped a factor of $1/R$ from all the expressions that follow for the I currents):

$$\begin{aligned}
I_h^{00} &= T^{++} + T^{+-} + (I_h^{00})_8 + \mathcal{O}(X^{12}) \\
&= \mathbf{STr} (\mathbf{1} + F^{0\mu} F_\mu^0 + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{O}(\Theta^2)) + (I_h^{00})_8 + \mathcal{O}(X^{12}).
\end{aligned} \tag{53}$$

All components of these matrix string currents are straightforward to write down, so we simply present them. For the NS-NS sector, the matrix string current components can be written in 10-dimensional notation as:

$$I_h^{00} = T^{++} + T^{+-} + (I_h^{00})_8 + \mathcal{O}(X^{12})$$

$$\begin{aligned}
&= \mathbf{STr}(\mathbb{1} + F^{0\mu}F_\mu^0 + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(\Theta^2)) + (I_h^{00})_8 + \mathcal{O}(X^{12}), \\
I_h^{0i} &= T^{+i} + T^{-i} + \mathcal{O}(X^{10}) \\
&= -\mathbf{STr}(F^{0i} + F^{0\mu}F_{\mu\nu}F^{\nu i} + \frac{1}{4}F^{0i}F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(\Theta^2, \Theta^4)) + \mathcal{O}(X^{10}), \\
I_h^{ij} &= T^{ij} + (I_h^{ij})_8 + \mathcal{O}(X^{12}) = \mathbf{STr}(F^{i\mu}F_\mu^j + \mathcal{O}(\Theta^2)) + (I_h^{ij})_8 + \mathcal{O}(X^{12}), \\
I_\phi &= T^{++} - \frac{1}{3}(T^{+-} + T^{ii}) + (I_\phi)_8 + \mathcal{O}(X^{12}) \\
&= \mathbf{STr}(\mathbb{1} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(\Theta^2)) + (I_\phi)_8 + \mathcal{O}(X^{12}), \\
I_s^{0i} &= 3J^{+-i} + \mathcal{O}(X^8) = \frac{1}{2}\mathbf{STr}(F^{0\mu}F_\mu^i + \mathcal{O}(\Theta^2)) + \mathcal{O}(X^8), \\
I_s^{ij} &= 3J^{+ij} - 3J^{-ij} + \mathcal{O}(X^{10}) \\
&= -\frac{1}{2}\mathbf{STr}(F^{ij} + F^{i\mu}F_{\mu\nu}F^{\nu j} + \frac{1}{4}F^{ij}F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(\Theta^2, \Theta^4)) + \mathcal{O}(X^{10}). \tag{54}
\end{aligned}$$

Moving to the R–R fields, one can write the matrix string currents in the 10–dimensional notation as:

$$\begin{aligned}
I_0^0 &= T^{++} = \mathbf{STr}(\mathbb{1}), \\
I_0^i &= T^{+i} = -\mathbf{STr}(F^{0i}), \\
I_2^{0ij} &= J^{+ij} + \mathcal{O}(X^{10}) = -\frac{1}{6}\mathbf{STr}(F^{ij}) + \mathcal{O}(X^{10}), \\
I_2^{ijk} &= J^{ijk} + \mathcal{O}(X^8) = \frac{1}{6}\mathbf{STr}(F^{0i}F^{jk} + F^{0j}F^{ki} + F^{0k}F^{ij} + \mathcal{O}(\Theta^2)) + \mathcal{O}(X^8), \\
I_4^{0ijkl} &= 6M^{+-ijkl} + \mathcal{O}(X^8) = \frac{1}{2}\mathbf{STr}(F^{ij}F^{kl} + F^{ik}F^{lj} + F^{il}F^{jk} + \mathcal{O}(\Theta^2)) + \mathcal{O}(X^8), \\
I_4^{ijklm} &= -6M^{-ijklm} + \mathcal{O}(X^{10}) = \frac{15}{2}\mathbf{STr}(F^{0[i}F^{jk}F^{lm]}) + \mathcal{O}(\Theta^2)) + \mathcal{O}(X^{10}), \\
I_6^{0ijklmn} &= S^{+ijklmn} + \mathcal{O}(X^{10}) = \mathbf{STr}(F_{[ij}F_{kl}F_{mn]}) + \mathcal{O}(X^{10}), \\
I_6^{ijklmnp} &= S^{ijklmnp} + \mathcal{O}(X^{12}) = 7\mathbf{STr}(F_{[ij}F_{kl}F_{mn}\dot{X}_{p]}) + \mathcal{O}(\theta^2, \theta^4)) + \mathcal{O}(X^{12}). \tag{55}
\end{aligned}$$

As we discussed in the previous section, finding a Matrix theory form for the transverse $M5$ –brane current components M^{+ijklm} and M^{ijklmn} is a matter which is yet not quite fully understood.

All these straighten out, we can proceed and explicitly write down (51) and (52) for this case of the IIB D –string. From the previous expressions, one obtains:

$$\begin{aligned}
S_{NS-NS}^{D1} &= (\phi - \frac{1}{2}h_a^a)\mathbf{STr}\left(\mathbb{1} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \dots\right) \\
&\quad + \frac{1}{2}h_{ij}\mathbf{STr}\left(F^{i\mu}F_\mu^j + \dots\right) - \frac{1}{2}h_{ab}\mathbf{STr}\left(F^{a\mu}F_\mu^b + \dots\right)
\end{aligned}$$

$$\begin{aligned}
& -h_{ai} \mathbf{STr} \left(F^{ai} + F^{a\mu} F_{\mu\nu} F^{\nu i} + \frac{1}{4} F^{ai} F_{\mu\nu} F^{\mu\nu} + \dots \right) \\
& -\frac{1}{2} B_{ij} \mathbf{STr} \left(F^{ij} + F^{i\mu} F_{\mu\nu} F^{\nu j} + \frac{1}{4} F^{ij} F_{\mu\nu} F^{\mu\nu} + \dots \right) \\
& +\frac{1}{2} B_{ab} \mathbf{STr} \left(F^{ab} + F^{a\mu} F_{\mu\nu} F^{\nu b} + \frac{1}{4} F^{ab} F_{\mu\nu} F^{\mu\nu} + \dots \right) \\
& +B_{ai} \mathbf{STr} \left(F^{a\mu} F_{\mu}{}^i + \dots \right) + \dots,
\end{aligned} \tag{56}$$

$$\begin{aligned}
S_{R-R}^{D1} &= \frac{1}{2} \epsilon^{ab} C^{D(-1)} \mathbf{STr} (F_{ab} + \dots) + \frac{1}{2} \epsilon^{ab} C_{ab}^{D1} \mathbf{STr} (\mathbb{1} + \dots) + \frac{1}{2} \epsilon^{ab} C_{ai}^{D1} \mathbf{STr} (F_b{}^i + \dots) \\
&+ \frac{1}{2} \epsilon^{ab} C_{ij}^{D1} \mathbf{STr} \left(F_a{}^i F_b{}^j - \frac{1}{2} F_{ab} F^{ij} + \dots \right) + \dots,
\end{aligned} \tag{57}$$

where in the previous expressions one should still take into consideration that the tensors must be appropriately reduced to 2-dimensional world-volume notation via the identifications $F_{ai} \equiv D_a X^i$ and $F_{ij} \equiv i[X^i, X^j]$, and the proper treatment of the compact coordinate X^9 according to [22]. Integration over the cylindrical world-sheet $\{\tau, \sigma\}$ is implicit. Of course the action (56) and (57) is quite interesting on its own, as it yields the gauged matrix sigma model for the Type IIB D -string in weakly curved backgrounds.

The issue of T -duality along R_9^{IA} solved, let us now deal with the IIB S -duality transformation. The $SL(2, \mathbf{Z})$ duality symmetry of Type IIB string theory maps a (p, q) -string into another (p', q') -string. In here, we shall focus on the usual \mathbf{Z}_2 subgroup of the S -duality group generated by the transformation which exchanges the NS-NS and R-R 2-form fields, and in particular maps the D -string into the fundamental string. As in the case of T -duality, the action of this subgroup of S -duality on arbitrary IIB supergravity background fields is well known [42]. At linear order the transformation is [35],

$$\begin{aligned}
\phi &\longrightarrow -\phi \\
C^{(0)} &\longrightarrow -C^{(0)} \\
B_{\mu\nu} &\longrightarrow -C_{\mu\nu}^{(2)} \\
C_{\mu\nu}^{(2)} &\longrightarrow B_{\mu\nu} \\
h_{\mu\nu} &\longrightarrow h_{\mu\nu} \\
C^{(4)} &\longrightarrow C^{(4)}.
\end{aligned} \tag{58}$$

These transformations are written in the Einstein frame, even though we have been working in the string frame. This is fine as the terms that we are considering from the string action, in this paper, are the same in both frames.

The problem we face concerns the implementation of S -duality at the D -string world-volume level. In fact, the S -duality transformation properties of the world-volume operators in the 2-dimensional $U(N)$ gauge theory are not known. What we shall see, is that in the

end the S -duality transformations are not so surprising, and they turn out to be quite simple as there will be no change in the composite operators. But for the moment, let us assume that they could be anything.

In here, for the $D1$ -brane, because we know what the result is from matrix string theory we can predict the precise transformation properties of all the operators that appear in the action. For the moment, we shall perform the IIB S -duality transformation of the D -string action (56) and (57) in the following way. We apply the linear S -duality transformations (58) to the background fields, and we denote S -duals of the world-volume fields by simply putting a tilde over them. Next, as we T -dualize back to the IIA theory, we can then compare to the previous section and obtain predictions for all these “tilded” operators. S -dualizing the D -string action in this way, one obtains:

$$\begin{aligned}
S_{NS-NS}^{F1} = & \left(-\phi - \frac{1}{2} h_a{}^a \right) \mathbf{STr} \left(\mathbb{1} + \frac{1}{4} F_{\mu\nu} \widetilde{F}^{\mu\nu} + \dots \right) \\
& + \frac{1}{2} h_{ij} \mathbf{STr} \left(F^{i\mu} \widetilde{F}_\mu{}^j + \dots \right) - \frac{1}{2} h_{ab} \mathbf{STr} \left(F^{a\mu} \widetilde{F}_\mu{}^b + \dots \right) \\
& - h_{ai} \mathbf{STr} \left(\widetilde{F}^{ai} + F^{a\mu} \widetilde{F}_{\mu\nu} F^{\nu i} + \frac{1}{4} F^{ai} \widetilde{F}_{\mu\nu} F^{\mu\nu} + \dots \right) \\
& + \frac{1}{2} C_{ij} \mathbf{STr} \left(\widetilde{F}^{ij} + F^{i\mu} \widetilde{F}_{\mu\nu} F^{\nu j} + \frac{1}{4} F^{ij} \widetilde{F}_{\mu\nu} F^{\mu\nu} + \dots \right) \\
& - \frac{1}{2} C_{ab} \mathbf{STr} \left(\widetilde{F}^{ab} + F^{a\mu} \widetilde{F}_{\mu\nu} F^{\nu b} + \frac{1}{4} F^{ab} \widetilde{F}_{\mu\nu} F^{\mu\nu} + \dots \right) \\
& - C_{ai} \mathbf{STr} \left(F^{a\mu} \widetilde{F}_\mu{}^i + \dots \right) + \dots,
\end{aligned} \tag{59}$$

$$\begin{aligned}
S_{R-R}^{F1} = & -\frac{1}{2} \epsilon^{ab} C^{D(-1)} \mathbf{STr} \left(\widetilde{F}_{ab} + \dots \right) + \frac{1}{2} \epsilon^{ab} B_{ab} \mathbf{STr} \left(\mathbb{1} + \dots \right) + \frac{1}{2} \epsilon^{ab} B_{ai} \mathbf{STr} \left(\widetilde{F}_b{}^i + \dots \right) \\
& + \frac{1}{2} \epsilon^{ab} B_{ij} \mathbf{STr} \left(F_a{}^i \widetilde{F}_b{}^j - \frac{1}{2} F_{ab} \widetilde{F}^{ij} + \dots \right) + \dots,
\end{aligned} \tag{60}$$

where we also denoted the action subscripts with a tilde, as we have a mixing of the NS-NS and R-R sectors under S -duality. A note on notation: in case it is not clear, in the previous expressions the tilde is over the whole composite operator.

Proceeding, we are left with a final T -duality along R_9^{IIB} that leads back to the Type IIA theory, and therefore to matrix string theory. The rules for T -duality have been previously explained and used, so we just apply them to the previous expressions to obtain the matrix string theory action in weak background fields. Two points should still be stressed. We have been slightly abusive of notation in the previous expressions by allowing more than one compact direction, as we indexed tensors as \hat{a}, \hat{b}, \dots . Of course in the case we are dealing with there is only one, σ . Moreover, we still have to write the world-volume tensors in the 2-dimensional world-sheet notation. Once we make the notation completely rigorous, we are left with (recall that in the matrix string limit $R_9 \rightarrow 0$),

$$\begin{aligned}
\mathcal{S} = & \phi \mathbf{STr} \left(-\mathbb{1} + \frac{1}{2} \widetilde{\dot{X}}_i^2 - \frac{1}{2} (D\widetilde{X}^i)^2 + \frac{1}{2} g_s^2 \widetilde{\dot{A}}^2 + \frac{1}{2g_s^2} \sum_{i<j} [X^i, \widetilde{X}^j]^2 + \dots \right) \\
& + \frac{1}{2} h_{00} \mathbf{STr} \left(\mathbb{1} + \frac{1}{2} \widetilde{\dot{X}}_i^2 + \frac{1}{2} (D\widetilde{X}^i)^2 + \frac{1}{2} g_s^2 \widetilde{\dot{A}}^2 - \frac{1}{2g_s^2} \sum_{i<j} [X^i, \widetilde{X}^j]^2 + \dots \right) \\
& + \frac{1}{2} h_{ij} \mathbf{STr} \left(\dot{X}^i \widetilde{\dot{X}}^j - (DX^i) \widetilde{(DX^j)} - \frac{1}{g_s^2} [X^i, X^k] \widetilde{[X^k, X^j]} + \dots \right) \\
& + h_{0i} \mathbf{STr} \left(\widetilde{\dot{X}}^i + \frac{1}{2} g_s^2 \widetilde{\dot{A}}^2 \widetilde{\dot{X}}^i - i \dot{A} (DX^j) [X^j, X^i] + \frac{1}{2} (\dot{X}^j)^2 \widetilde{\dot{X}}^i - \dot{X}^j (D\widetilde{X}^j) (DX^i) \right. \\
& + \frac{1}{2} (D\widetilde{X}^j)^2 \widetilde{\dot{X}}^i - \frac{1}{g_s^2} \dot{X}^j [X^j, \widetilde{X}^k] [X^k, X^i] - \frac{1}{2g_s^2} \dot{X}^i \sum_{j<k} [\widetilde{X}^j, X^k]^2 + \dots \left. \right) \\
& + B_{0i} \mathbf{STr} \left(-\frac{1}{2} (D\widetilde{X}^i) + \dots \right) + B_{ij} \mathbf{STr} \left(-\frac{1}{2} \dot{X}^i (\widetilde{DX}^j) + \frac{1}{2} \dot{X}^j (\widetilde{DX}^i) - \frac{i}{2} \dot{A} [\widetilde{X}^i, X^j] \right. \\
& + \dots \left. \right) + C_i \mathbf{STr} \left(g_s \widetilde{\dot{A}} \widetilde{\dot{X}}^i - \frac{i}{g_s} (DX^j) \widetilde{[X^j, X^i]} + \dots \right) + C_0 \mathbf{STr} \left(g_s \widetilde{\dot{A}} + \frac{1}{2} g_s^3 \widetilde{\dot{A}}^3 \right. \\
& - \frac{1}{2} g_s \dot{A} (\widetilde{DX}^i)^2 + \frac{1}{2} g_s \dot{A} (\widetilde{\dot{X}}^i)^2 + \frac{i}{g_s} \dot{X}^i [X^i, \widetilde{X}^j] DX^j - \frac{1}{2g_s} \dot{A} \sum_{i<j} [\widetilde{X}^i, X^j]^2 + \dots \left. \right) \\
& + \dots
\end{aligned} \tag{61}$$

This completes the sequence of DVV dualities. The final result should be equivalent to performing the 9 – 11 flip of section 4. Comparing this result to the one in section 4, (47), one will obtain the S -duality transformation rules for the tensor operators that live on the D -string world-volume. One should only keep in mind that in this section we have dropped a factor of $1/R$ from all the expressions, and that one may need to correct for units as comparing to the previous section.

From the expression (61) we have results for the I_ϕ , I_h^{00} , I_h^{ij} , I_h^{0i} , I_s^{0i} , I_s^{ij} , I_0^i and I_0^0 tensors coming from the T - S - T chain of dualities. It should be a straightforward exercise to include all other matrix string tensors in this result. For our purposes this is enough. Comparing back to what we have obtained for those same tensors in section 4 – where we used the 9 – 11 flip to DVV reduce – we obtain an interesting result: the 2-dimensional D -string world-volume composite operators are invariant under the target space IIB S -duality operation. It is indeed somewhat expected that these operators should not change, as from the field theory side we expect non-trivial S -duality properties only for the $\mathcal{N} = 4$, $d = 4$ gauge theory, *i.e.*, we only expect to see non-trivial transformation laws for the operators that live on the world-volume of the $D3$ -brane. For the $D1$ -brane, the operators are kept fixed under the target transformation.

6 Green–Schwarz Action in a Curved Background

Given that the matrix string theory action we have built has been firmly established at the level of string duality, we would further like to confirm it by looking at its conformal field theory limit, where we obtain the free string case. We expect that when we take the $g_s \rightarrow 0$ limit, our action should match the Green–Schwarz sigma model for the Type IIA superstring [36], once we consider this latter one in the same weak background approximation that we are considering in here.

6.1 The Green–Schwarz Action

We begin with a short review of the results obtained for the IIA superstring in [36], so that we can establish a bridge between the results of that paper and our notation. We want to compare the Green–Schwarz sigma model to our matrix sigma model, and for that all one needs to do is to consider the matrix sigma model in the free string limit which was presented throughout section 4 for all the tensor fields. One should also take into consideration the weak field approximation in the sigma model for one string. It should be stressed that we shall be looking at schematic and qualitative results only, throughout this section. This is because establishing a precise map from the matrix string theory in the IR to the light–cone Green–Schwarz action requires a precise lifting of the IR matrix action from the cylinder to its branched coverings, and so to any given Riemann surface. The procedure is described at length in [24, 25, 26, 27, 28] for the flat background situation. Completing such a procedure for this curved situation is an interesting project for the future.

The massless spectra of the Type IIA closed superstring includes the metric, $g_{\mu\nu}$, the NS B -field, $B_{\mu\nu}$, and the dilaton, ϕ , whose origin is the bosonic sector of the superstring action. It also includes the $D0$ -brane 1-form, C_μ , and the $D2$ -brane 3-form, $C_{\mu\nu\rho}$, whose origin is the fermionic sector of the same superstring action. The covariant superstring action can be written while in the presence of couplings to the background fields of $\mathcal{N} = 2$ 10-dimensional supergravity, as was shown in [36]. In here we are interested in the form of this action in light–cone gauge, which was also derived in [36]: If one chooses light–cone gauge, and furthermore assumes the supergravity background fields to be non-trivial only in the eight transverse directions (so that the background spacetimes decomposes as $\mathcal{M}^{10} = \mathbf{R}^{(1,1)} \times \mathcal{M}^8$), then the NS–NS sector of the IIA superstring action is written as [36],

$$\begin{aligned} \mathcal{L}_{NS-NS} = & g_{ij}(X) \sigma^{ab} \partial_a X^i \partial_b X^j + 4\pi\alpha' B_{ij}(X) \epsilon^{ab} \partial_a X^i \partial_b X^j - 2i\theta^T \gamma^0 \rho^{\hat{0}} \gamma^- \rho^a \widehat{D}_a \theta \\ & + \frac{1}{64} \widehat{R}_{ijkl} \theta^T \gamma^0 \rho^{\hat{0}} \gamma^{ij-} \rho^a (1 + \rho_3) \theta \theta^T \gamma^0 \rho^{\hat{0}} \gamma^{kl-} \rho^a (1 - \rho_3) \theta + \dots, \end{aligned} \quad (62)$$

where we have the following relations,

$$\widehat{R}^i_{jkl} = \partial_k \widehat{\Gamma}^i_{jl} - \dots, \quad \widehat{\Gamma}^i_{jl} = \Gamma^i_{jl}[g] + 2\pi\alpha' H^i_{jl}, \quad H_{ijk} = 3\partial_{[i} B_{jk]},$$

$$\widehat{D}_a = \partial_a - \frac{1}{4} \gamma^{\hat{j}\hat{k}} \widehat{\omega}_i^{\hat{j}\hat{k}} \partial_a X^i, \quad \widehat{\omega}_{..i} = \omega_{..i} + 2\pi\alpha' \rho_3 H_{..i}, \quad \gamma^{ij-} \equiv \gamma^{ij} \gamma^-, \quad (63)$$

and moreover: i, j, k, \dots , are transverse spacetime indices; a, b, \dots are world-sheet indices; hatted indices correspond to tangent frame indices; σ_{ab} is the world-sheet metric; and we have introduced two-dimensional world-sheet Dirac matrices $\rho_{\hat{a}}$ as,

$$\rho^{\hat{0}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^{\hat{1}} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (64)$$

One can also write down the Lagrangian for the R-R sector of the Type IIA superstring action [36],

$$\begin{aligned} \mathcal{L}_{R-R} = & \frac{i}{(\alpha')^{3/2}} \theta^T \gamma^0 \rho^{\hat{0}} \gamma^{(i} \Gamma^\Lambda \gamma^{j)} (1 - \rho_3) \theta (-\partial^a X^i \partial_a X^j + \frac{i}{6} \theta^T \gamma^0 \rho^{\hat{0}} \gamma^{ni-} \rho^a \theta \partial_a X^j \partial_n \\ & - \frac{1}{144} \theta^T \gamma^0 \rho^{\hat{0}} \gamma^{ni-} \rho^a \theta \theta^T \gamma^0 \rho^{\hat{0}} \gamma^{mj-} \rho_a \theta \partial_n \partial_m) C_\Lambda(X) \\ & + \frac{i}{(\alpha')^{3/2}} \theta^T \gamma^0 \rho^{\hat{0}} \gamma^{[i} \Gamma^\Lambda \gamma^{j]} (1 - \rho_3) \theta (-\epsilon^{ab} \partial_a X^i \partial_b X^j + \frac{i}{6} \theta^T \gamma^0 \rho^{\hat{0}} \gamma^{ni-} \rho^a \rho_3 \theta \partial_a X^j \partial_n \\ & - \frac{1}{144} \theta^T \gamma^0 \rho^{\hat{0}} \gamma^{ni-} \rho^a \theta \theta^T \gamma^0 \rho^{\hat{0}} \gamma^{mj-} \rho_a \rho_3 \theta \partial_n \partial_m) C_\Lambda(X) + \dots, \end{aligned} \quad (65)$$

where we have $\{(\Gamma^\Lambda, C_\Lambda)\} = \{(\gamma^{-i}, C_i), (\gamma^{ijk-}, C_{ijk})\}$ for the background $D0$ and $D2$ -brane currents. Indices i, j, \dots , are contracted via the metric g_{ij} , and the dots in (65) refer to higher-derivative terms (which have no contribution to the light-cone vertex operators) [36]. This completes the information on the Green-Schwarz action that we shall be interested in.

To begin the comparison with the abelian limit of our results from section 4, we look at the NS-NS sector. In the weak background field limit we are considering, the Riemann curvature terms drops out from (62) and all we need to check is for the existence of the abelian couplings,

$$h_{ij}(X) \partial^a X^i \partial_a X^j, \quad (66)$$

and

$$B_{ij}(X) \epsilon^{ab} \partial_a X^i \partial_b X^j. \quad (67)$$

Comparing back to (34) and (38), one immediately observes that these terms indeed appear in the abelian conformal limit of our matrix string action. The term involving the Riemann tensor is present in a weak field approximation only via its piece in $\partial^2 h$. One can realize however that, being schematically of the form $\widehat{R}_{ijkl} \theta^T \Gamma^{ij} \theta \theta^T \Gamma^{kl} \theta$, it is of order $\mathcal{O} = (\frac{R}{\ell_s})^2$. At the Matrix theory level we have four fermion terms of this type in the T^{-i} component of the Matrix stress tensor and in the J^{-ij} component of the Matrix membrane current, and therefore we have terms of this type in several components of the matrix string tensor couplings. However, none of these terms seems to have the required index structure to couple

to the Riemann curvature term, and besides we are looking for tensors that will couple to a term of the type $\partial_i \partial_j h_{kl}$. The reason for this is that such couplings will actually arise from higher moment terms. Indeed, one can see from the appendix that there are two fermion contributions to the first moment terms of $T_{\text{Fermion}}^{ij(l)}$, which will couple to a term in ∂h . Similarly there will be four fermion contributions that should produce the required curvature coupling. It would be interesting to construct explicitly such terms.

When we move to the R–R sector, we observe that in (65) the terms that do not involve derivatives of the $D0$ and $D2$ -brane fields are at order $\mathcal{O} = (\frac{R}{\ell_s})^2$, and schematically of the form $\theta^T \Gamma_{ij\Lambda} \theta \partial X^i \partial X^j$. For the $D0$ -brane such terms will likely come from the quadratic fermion pieces that we neglected in the tensor T^{--} , while for the $D2$ -brane case it is not entirely clear where these terms should come from. The other terms in the R–R action (65) are higher derivative terms in the R–R fields, and so we would only expect to match these terms to higher moments of our couplings. Therefore, the overall comparison of our results with the ones for the one string action is rather schematic and qualitative. But the comparison can still be of some use in predicting some possible new coupling terms coming up in the full curved action.

6.2 Matrix Theory in Curved Backgrounds

As we mentioned before, in the Green–Schwarz sigma model there are four fermion couplings to the background Riemann tensor. One could expect that this coupling would correspond to the free string limit of some nonabelian coupling between world-sheet fields and the background curvature. Indeed, given the previous match between abelian and nonabelian actions, one has some clues for the form of the couplings to background curvature. This would amount to terms in the full non-linear matrix string action, and therefore to terms involving the Riemann curvature and other non-linear combinations of the supergravity background fields in the Matrix theory action.

As we have just seen, the term involving the curvature tensor is schematically of the form $\hat{R}_{ijkl} \theta^T \Gamma^{ij} \theta \theta^T \Gamma^{kl} \theta$, and we have no tensor coupling with this index structure among the zeroeth moment terms. One naturally expects that such tensors would start making their appearance once one performs a higher loop calculation in Matrix theory or in matrix string theory as we would obtain, *e.g.*, quadratic pieces in the metric, h . For the moment we simply observe that the actions (62) and (65) yield already some information on what one will obtain from such a higher loop calculation by telling us what will be the abelian limit of the tensors one would eventually obtain. Indeed we can predict that there will be a tensor coupling to the quadratic metric piece, of the type,

$$G^{ijkl} = \theta^T \Gamma^{ij} \theta \theta^T \Gamma^{kl} \theta + \text{Nonabelian Terms.} \quad (68)$$

This is of course the required coupling for the curved matrix string theory action to match, in the free abelian limit, the Green–Schwarz action. But because we are dealing in this section with background fields that are non-trivial only in the eight transverse directions,

this term actually is lifted to a term in the curved Matrix theory action. We see therefore that in order for a supersymmetric completion of the curved background Matrix action to be done, there will be at least quartic fermion terms required, at zeroeth moment terms.

Because the curvature term in (62) also includes a coupling to the NS–NS B -field field strength, a similar story will also work for that term. Likewise, due to the higher derivative terms in (65), we shall actually obtain quite a few predictions for higher moments and other tensor couplings, along the lines of the previous discussion for the background Riemann curvature. In summary, a full extension of Matrix theory for the case of curved backgrounds will probably not be as simple as the sigma model proposal in [15]. Its supersymmetric completion however, will have to include a series of non-linear background couplings, as we have just discussed. Such types of couplings, involving the Riemann tensor and four fermion fields, are common in supersymmetric completions of bosonic sigma models and so are quite natural to be expected in here as well.

7 Noncommutative Backgrounds

In this section we wish to exemplify the nonabelian nature of our action. Ideally one hopes that coherent states of gravitons can be made out of many fundamental strings (by some sort of fundamental string condensation), and by using infinite dimensional matrices to describe such solutions one could then be able to build fully curved spacetime geometries – somewhat like when one uses infinite matrices to describe non-compact curved membranes in Matrix theory [43]. In practice the situation is not as idealistic, as it is not clear how to go from the description of the coherent state in terms of the strings to their effects on the other strings. This would correspond to a higher loop calculation in Matrix theory or matrix string theory.

Still, an interesting question is whether one can exponentiate the noncommutative vertex operators we have obtained in our linear action, and from that build the full non-linear matrix sigma model. Recall that the results for the I tensors can be used (loosely speaking, via multiplication by $\exp(ip \cdot X)$) to obtain the noncommutative vertex operators of matrix string theory. However, precisely because of this noncommutative nature of the vertex operators, one still lacks an ordering for the exponentiation. Such an ordering would moreover have to produce terms with derivatives of the background fields, as such terms are expected in order to satisfy the geodesic length condition in [44, 45]. It is certainly not clear how to choose the ordering of such an exponential at this stage. There is also the question of whether the background satisfies the equations of motion of supergravity. To clear this issue, one would again need the full matrix sigma model in order to compute noncommutative beta functions and from there derive the noncommutative equations of motion for the background.

For the moment we will aim lower and consider a very simple example involving non-trivial R–R flux. There is a particular interest in examples involving R–R flux, due to its possible connection to noncommutative spacetime geometry. Recently there has been some study in the applications of noncommutative geometry to string theory. This has, however,

been mainly studied at the world-volume level where the noncommutativity appears as a result of non-trivial NS-NS flux [37, 38, 39, 40]. But it has also been suggested that in situations involving R-R flux rather than NS-NS flux, the noncommutativity could make its appearance at the background spacetime level due to small distance stringy effects [46]. It would be quite interesting to further study this issue.

7.1 R-R Flux and String Condensation

An interesting situation is the one where there exists non-trivial R-R flux. In recent work [41, 47] it was studied an example where a collection of $D0$ -branes was polarized into a noncommutative 2-sphere configuration by an external R-R field. The question quickly arises of whether a similar situation could exist in this case, where we are dealing with a collection of fundamental strings. If they can indeed be polarized into noncommutative configurations by some external R-R fields, this would then correspond to the creation of some sort of noncommutative stringy object. We shall see that such indeed happens, and so R-R flux can act as a source for fundamental string world-sheet noncommutativity.

Let us consider a situation where there is non-trivial R-R 3-form flux. This case will be quite similar to the one of dielectric branes considered in [41], the difference being that now we have “dielectric strings”. We will moreover consider a simplified case where we take all fermionic fields to vanish, $\theta = 0$. As an ansatz, let us also consider $A = 0$ and $\partial X^i = 0$. At the background level, we shall set all other fields (except for the membrane current) to zero. All this done, one is left with the flat space matrix string theory action,

$$S_{Flat} = \frac{1}{2\pi} \int d\sigma d\tau \, \text{Tr} \left(\frac{1}{2} (\dot{X}^i)^2 + \frac{1}{4g_s^2} [X^i, X^j]^2 \right), \quad (69)$$

supplemented by the $D2$ -brane linear coupling,

$$S_{D2-brane} = \frac{1}{2\pi} \int d\sigma d\tau \, C_{\mu\nu\lambda}(X) I_2^{\mu\nu\lambda}. \quad (70)$$

As to the $D2$ -brane linear coupling, we shall focus our attention in the lowest order terms in the derivative expansion. In particular, we will not retain terms at order $\mathcal{O} = (\frac{R}{\ell_s})$ and above (this corresponds to two operator insertions and is the same order as the flat space matrix string action). In this case, the tensor couplings we need to consider are:

$$\begin{aligned} I_2^{0ij} &= \text{STr} \left(-\frac{i}{6g_s} \left(\frac{\ell_s}{R} \right) [X^i, X^j] + \mathcal{O}\left(\frac{R}{\ell_s}\right) \right), \\ I_2^{ijk} &= \text{STr} \left(-\frac{i}{6g_s} \dot{X}^i [X^j, X^k] - \frac{i}{6g_s} \dot{X}^j [X^k, X^i] - \frac{i}{6g_s} \dot{X}^k [X^i, X^j] \right), \end{aligned} \quad (71)$$

so that the $D2$ -brane linear coupling term becomes,

$$\begin{aligned}
\frac{1}{2\pi} \int d\sigma d\tau C_{\mu\nu\lambda}(X) I_2^{\mu\nu\lambda} &= \frac{1}{2\pi} \int d\sigma d\tau \left(3C_{0ij}(X) I_2^{0ij} + C_{ijk}(X) I_2^{ijk} \right) \\
&= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{i}{2g_s} \left(\frac{\ell_s}{R} \right) C_{0ij}(X) [X^i, X^j] \right. \\
&\quad \left. - \frac{i}{2g_s} C_{ijk}(X) \dot{X}^i [X^j, X^k] + \mathcal{O}\left(\frac{R}{\ell_s}\right) \right), \tag{72}
\end{aligned}$$

where we still have to expand C_{0ij} to first order in derivatives, $C_{0ij}^{D2}(X) = C_{0ij}^{D2}(0) + (\frac{R}{\ell_s}) X^k \partial_k C_{0ij}^{D2}(0) + \dots$. Doing this and integrating the time derivative by parts, one obtains,

$$\frac{1}{2\pi} \int d\sigma d\tau C_{\mu\nu\lambda}(X) I_2^{\mu\nu\lambda} = \frac{1}{2\pi} \int d\sigma d\tau \frac{i}{2g_s} \mathbf{STr} \left((\partial_0 C_{ijk} - \partial_k C_{0ij}) X^k [X^i, X^j] \right) + \dots, \tag{73}$$

where we have further specialized for the particular case of time independent solutions, $\dot{X}^i = 0$. Any solutions we shall obtain will correspond to static backgrounds. Now, the $D2$ -brane 3-form potential, \mathbf{C}_3 , is related to the $D2$ -brane 4-form field strength, \mathbf{F}_4 , by the standard relation $\mathbf{F}_4 = d\mathbf{C}_3$, and so one simply has the expected gauge invariant coupling:

$$\frac{1}{2\pi} \int d\sigma d\tau C_{\mu\nu\lambda}(X) I_2^{\mu\nu\lambda} = \frac{1}{2\pi} \int d\sigma d\tau \frac{i}{6g_s} \mathbf{STr} \left(F_{0ijk} X^k [X^i, X^j] \right) + \dots \tag{74}$$

If we furthermore choose the $D2$ -brane field strength as constant, $F_{0ijk} = -2F\epsilon_{ijk}$ for $i, j, k = 1, 2, 3$, and zero otherwise, we are led into a situation very similar to the $D0$ -brane case of [41], only now with F -strings. Indeed, the effective potential we have obtained for the static configuration of N fundamental strings is,

$$V_{eff}(X) = -\frac{1}{4g_s^2} \mathbf{Tr} [X^i, X^j]^2 - \frac{i}{6g_s} F_{0ijk} \mathbf{STr} [X^i, X^j] X^k. \tag{75}$$

with the following equation for the extrema,

$$[[X^i, X^j], X^j] - \frac{i}{2} g_s F_{0ijk} [X^j, X^k] = 0. \tag{76}$$

The case of commuting matrices, $[X^i, X^j] = 0$, is a solution with zero potential energy. This corresponds to the free string limit, *i.e.*, it corresponds to a situation describing separated, straight and static free strings. More interesting to us would be a noncommuting solution. Following [48, 41], we consider the ansatz $X^i = \varphi \sigma^i$, $i = 1, 2, 3$, where φ is a constant and σ^i are some N -dimensional matrix representation of the $su(2)$ algebra,

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk} \sigma^k. \tag{77}$$

Using this ansatz in the equation for the extrema of the string effective potential, one immediately obtains that this is indeed a solution once we set the constant to be $\varphi = \frac{1}{2} g_s F$. If

we moreover consider the σ^i as an irreducible representation, one computes the Casimir as $\text{Tr}(\sigma^i)^2 = \frac{1}{3}N(N^2 - 1)$, and so the nonabelian solution has an effective potential of,

$$V_{NC} = -\frac{g_s^2 F^4}{48}N(N^2 - 1), \quad (78)$$

i.e., the noncommutative solution has an energy lower than the commuting one. This means of course that the configuration corresponding to separated static free strings is unstable, and the strings will actually condense into a noncommutative solution. This solution is the noncommutative sphere, with radius $R = \frac{1}{2}g_s F N(1 - \frac{1}{N^2})^{\frac{1}{2}}$, as can be seen from the algebra (77). So, in conclusion, the presence of an R–R field has condensed the initially free fundamental strings into a static noncommutative spherical configuration. Also, we should point out that this solution corresponds to a string theory derivation of the commutation relations for fundamental strings in the presence of R–R fields, proposed in [46].

This phenomena leads to an interesting question. In the D –brane situation, the Myers’ effect [41] tells us that an external R–R field can polarize a collection of D –branes into a noncommutative configuration. The noncommutativity is present at the world–volume level, such that there is a background commutative spacetime with a noncommutative object made out of many D –branes inside. In the situation described in this section we are dealing with F –strings. So, even though the situation seems quite analogous to the one of the brane system, one also needs to take into account the fact that the F –strings actually describe gravitons, and we are thus led to a situation where the R–R field is building a noncommutative object made out of many gravitons. Now, graviton states generally correspond to curved spacetimes, however describing small fluctuations such as gravitational waves with small amplitudes. If one wishes to describe large fluctuations one needs to consider states with a semiclassical behavior which would correspond to coherent states – where one would run into the mentioned problems concerning exponentiation of nonabelian couplings.

The question still remains on how to interpret a noncommutative object made out of many gravitons, inside a background commutative spacetime. One speculative possibility is that this could actually correspond to a noncommutative background spacetime geometry. Indeed, one could imagine that if we put enough gravitons together, the noncommutative spherical configuration will grow up to a stage where its curvature is actually weak. Then, we would be in a position to expect that this large sphere would take up the role of the background spacetime, *i.e.*, the R–R flux we considered would have created some sort of noncommutative background spacetime geometry. It would be quite interesting to further study more complex examples of such situations.

8 Conclusions

We have seen in this paper how to construct an action for matrix string theory in weakly curved background fields. In the process, we have also studied its relation to T and S dualities. Such an action provides working ground to study multiple interacting strings in

both NS–NS and R–R backgrounds. For the particular case of an R–R background, we have seen how fundamental strings can condense into a nonabelian configuration thus building a noncommutative stringy configuration. The action we derived also allows for some discussion on how background non–linear curvature terms will couple to the Matrix theory action in a general curved background. With all this in hand, we believe there are quite a few interesting lines for future research on this subject. We present some of these lines below.

One possible application of the action we have obtained is to describe second quantized superstring theory in general backgrounds with R–R fields turned on. This is quite an interesting venue of work, given that there has been some recent interest in such ideas, *e.g.* [49, 50, 51, 52, 53, 54]. Along these lines, one should also try to further understand the possible background geometry noncommutativity due to the R–R flux, in particular it would be quite interesting if some connection to the work in [55] could be done. For this, it could be of some interest to complete further examples involving diverse background fields, in order to fully explore the nonabelian aspects of curved stringy spacetimes. This could be a first step towards the more ambitious goal of constructing fully curved noncommutative spacetime geometries from string theory.

The resulting action from our work could also be of use in the study of diverse scattering or absorption problems, involving branes, black holes, or in the context of the AdS/CFT correspondence [56, 57, 58]. It would be quite interesting if some work along the lines in [59] could be accomplished. Probably, from the computation of scattering amplitudes, the role of the noncommutative vertex operators would then become more translucent (there has been some very recent work on vertex operators in [60]). This would then be of some interest should it yield further insight on how one should exponentiate such operators, in order to obtain the full non–linear action. Indeed, one particularly interesting use of our work would be to further use this action in order to try to infer some new information about Matrix theory in a general curved background. This could perhaps be accomplished via a more detailed and direct comparison with the Type IIA string theory abelian limit. Some work towards such goal has been done in [15, 44, 45, 61, 62, 63, 64, 65, 66], and one should try to understand any possible relations between those papers and the work presented in here. Probably one particularly important relation to understand is the one with the work in [65]. In order to achieve these goals, some research should be done on understanding how to construct the precise lift of the matrix string theory action from the cylinder to arbitrary Riemann surfaces, and so establish the precise connection to the Green–Schwarz action along the lines of [24, 25, 26]. We hope to address some of these questions in the future.

Acknowledgments: I would like to thank Washington Taylor for many discussions and critical comments as well as suggestions on the draft of the paper. I would also like to thank Lorenzo Cornalba, José Mourão and João Nunes for helpful discussions and/or comments. The author is supported in part by the Fundação para a Ciência e Tecnologia, under the grant Praxis XXI BPD-17225/98 (Portugal).

A Supercurrents from Matrix Theory

We reproduce here the Matrix theory forms of the multipole moments of the 11-dimensional supercurrents found in [13, 14], written in 10-dimensional form as in [35]. Dropping a factor of $1/R$ from each expression, the stress tensor T^{IJ} , $M2$ -brane current J^{IJK} and $M5$ -brane current M^{IJKLMN} have integrated (monopole) components:

$$\begin{aligned}
T^{++} &= \mathbf{STr}(\mathbb{1}) = N \\
T^{+i} &= -\mathbf{STr}(F^{0i}) \\
T^{+-} &= \mathbf{STr}(F^{0\mu}F^0{}_{\mu} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\Theta}\Gamma^0 D_0\Theta) \\
T^{ij} &= \mathbf{STr}(F^{i\mu}F^j{}_{\mu} + \frac{i}{4}\bar{\Theta}\Gamma^i D_j\Theta + \frac{i}{4}\bar{\Theta}\Gamma^j D_i\Theta) \\
T^{-i} &= -\mathbf{STr}(F^{0\mu}F_{\mu\nu}F^{\nu i} + \frac{1}{4}F^{0i}F_{\mu\nu}F^{\mu\nu} - \frac{i}{8}F_{\mu\nu}\bar{\Theta}\Gamma^i\Gamma^{\mu\nu}D_0\Theta \\
&\quad + \frac{i}{8}F_{\mu\nu}\bar{\Theta}\Gamma^0\Gamma^{\mu\nu}D_i\Theta - \frac{i}{4}F_{\mu\nu}\bar{\Theta}\Gamma^{\nu}\Gamma^{0i}D^{\mu}\Theta - \frac{1}{8}\bar{\Theta}\Gamma^{0\mu i}\Theta\bar{\Theta}\Gamma_{\mu}\Theta) \\
T^{--} &= \frac{1}{4}\mathbf{STr}(F_{\mu\nu}F^{\nu\gamma}F_{\gamma\delta}F^{\delta\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}F_{\gamma\delta}F^{\gamma\delta} + iF_{\mu\nu}F_{\gamma\delta}\bar{\Theta}\Gamma^{\nu}\Gamma^{\gamma\delta}D^{\mu}\Theta + \mathcal{O}(\Theta^4)) \\
J^{+ij} &= -\frac{1}{6}\mathbf{STr}(F^{ij}) \\
J^{+-i} &= \frac{1}{6}\mathbf{STr}(F^{0\mu}F^i{}_{\mu} + \frac{i}{4}\bar{\Theta}\Gamma^0 D_i\Theta - \frac{i}{4}\bar{\Theta}\Gamma^i D_0\Theta) \\
J^{ijk} &= \frac{1}{6}\mathbf{STr}(F^{0i}F^{jk} + F^{0j}F^{ki} + F^{0k}F^{ij} - \frac{3i}{4}\bar{\Theta}\Gamma^{0[ij}D^{k]}\Theta + \frac{i}{4}\bar{\Theta}\Gamma^{ijk}D_0\Theta) \\
J^{-ij} &= \frac{1}{6}\mathbf{STr}(F^{i\mu}F_{\mu\nu}F^{\nu j} + \frac{1}{4}F^{ij}F_{\mu\nu}F^{\mu\nu} - \frac{i}{8}F_{\mu\nu}\bar{\Theta}\Gamma^j\Gamma^{\mu\nu}D_i\Theta \\
&\quad + \frac{i}{8}F_{\mu\nu}\bar{\Theta}\Gamma^i\Gamma^{\mu\nu}D_j\Theta - \frac{i}{4}F_{\mu\nu}\bar{\Theta}\Gamma^{\nu}\Gamma^{ij}D^{\mu}\Theta + \frac{1}{8}\bar{\Theta}\Gamma^{\mu ij}\Theta\bar{\Theta}\Gamma_{\mu}\Theta) \\
M^{+-ijkl} &= \frac{1}{12}\mathbf{STr}(F^{ij}F^{kl} + F^{ik}F^{lj} + F^{il}F^{jk} - i\bar{\Theta}\Gamma^{[ijk}D^{l]}\Theta) \\
M^{-ijklm} &= -\frac{5}{4}\mathbf{STr}(F^{0[i}F^{jk}F^{lm]}) + \frac{i}{2}F^{[0i}\bar{\Theta}\Gamma^{jkl}D^{m]}\Theta).
\end{aligned}$$

Here, \mathbf{STr} denotes a symmetrized trace in which one takes the average over all possible orderings of the matrices inside the trace, with commutators being treated as a unit block. Time derivatives are taken with respect to Minkowski time t . Indices i, j, \dots , run from 1 through 9, while indices a, b, \dots , run from 0 through 9. In these expressions one should use the definitions $F_{0i} = \dot{X}^i$ and $F_{ij} = i[X^i, X^j]$. A Matrix form for the transverse 5-brane current components M^{+ijklm} and M^{ijklmn} is not known, and in fact comparison with supergravity suggests that these should be zero for any Matrix theory configuration.

The higher multipole moments of these currents contain one set of terms which are found by including the matrices X^{k_1}, \dots, X^{k_n} into the symmetrized trace as well as more complicated spin contributions. We may write these as,

$$\begin{aligned}
T^{IJ(i_1 \dots i_k)} &= \text{Sym}(T^{IJ}; X^{i_1}, \dots, X^{i_k}) + T_{\text{Fermion}}^{IJ(i_1 \dots i_k)} \\
J^{IJK(i_1 \dots i_k)} &= \text{Sym}(J^{IJK}; X^{i_1}, \dots, X^{i_k}) + J_{\text{Fermion}}^{IJK(i_1 \dots i_k)} \\
M^{IJKLMN(i_1 \dots i_k)} &= \text{Sym}(M^{IJKLMN}; X^{i_1}, \dots, X^{i_k}) + M_{\text{Fermion}}^{IJKLMN(i_1 \dots i_k)},
\end{aligned}$$

where some simple examples of the two-fermion contribution to the first moment terms are,

$$\begin{aligned}
T_{\text{Fermion}}^{+i(j)} &= -\frac{1}{8R} \text{Tr}(\bar{\Theta} \Gamma^{[0ij]} \Theta) \\
T_{\text{Fermion}}^{+-i(j)} &= -\frac{1}{16R} \text{Tr}(F_{\mu\nu} \bar{\Theta} \gamma^{[\mu\nu i]} \Theta - 4 \bar{\Theta} F_{0\mu} \gamma^{[0\mu i]} \Theta) \\
T_{\text{Fermion}}^{ij(l)} &= \frac{1}{8R} \text{Tr}(F_{j\mu} \bar{\Theta} \gamma^{[\mu il]} \Theta + \bar{\Theta} F_{i\mu} \gamma^{[\mu jl]} \Theta) \\
J_{\text{Fermion}}^{+ij(k)} &= \frac{i}{48R} \text{Tr}(\bar{\Theta} \Gamma^{[ijk]} \Theta) \\
J_{\text{Fermion}}^{+-ij(j)} &= \frac{1}{48R} \text{Tr}(F_{0\mu} \bar{\Theta} \gamma^{[\mu ij]} \Theta + \bar{\Theta} F_{i\mu} \gamma^{[\mu 0j]} \Theta) \\
M_{\text{Fermion}}^{+-ijkl(m)} &= -\frac{i}{16R} \text{STr}(\bar{\Theta} F^{[jk} \Gamma^{il]m} \Theta).
\end{aligned}$$

The remaining two-fermion contributions to the first moments and some four-fermion terms are also determined by the results in [14]. There are also fermionic components of the supercurrent which couple to background fermion fields in the supergravity theory. These couplings have not been discussed in this paper, but the Matrix theory form of the currents is determined in [14].

Finally, there is also a 6-brane current appearing in Matrix theory related to nontrivial 11-dimensional background metrics. The components of this current as well as its first moments are:

$$\begin{aligned}
S^{+ijklmn} &= \frac{1}{R} \text{STr}(F_{[ij} F_{kl} F_{mn]}) \\
S^{+ijklmn(p)} &= \frac{1}{R} \text{STr}(F_{[ij} F_{kl} F_{mn]} X_p - \theta F_{[kl} F_{mn} \gamma_{pqr]} \theta) \\
S^{ijklmnp} &= \frac{7}{R} \text{STr}(F_{[ij} F_{kl} F_{mn} \dot{X}_{p]} + \mathcal{O}(\theta^2, \theta^4)) \\
S^{ijklmnp(q)} &= \frac{7}{R} \text{STr}\left(F_{[ij} F_{kl} F_{mn} \dot{X}_{p]} X_q - \theta \dot{X}_{[j} F_{kl} F_{mn} \gamma_{pqr]} \theta + \frac{i}{2} \theta F_{[jk} F_{lm} F_{np} \gamma_{qr]} \theta\right).
\end{aligned}$$

B DVV Reduction of Matrix Theory Tensors

The Matrix stress tensor components are:

$$\begin{aligned}
T^{++} &= \frac{1}{R} \mathbf{STr} (\mathbb{1}), \\
T^{+i} &= \frac{1}{R} \mathbf{STr} (\dot{X}_i), \\
T^{+-} &= \mathbf{STr} \left(\frac{1}{2R} \dot{X}_i \dot{X}_i - \frac{RM_P^6}{8\pi^2} \sum_{i < j} [X^i, X^j]^2 + \frac{RM_P^6}{8\pi^2} \theta \gamma^i [X^i, \theta] \right), \\
T^{ij} &= \mathbf{STr} \left(\frac{1}{R} \dot{X}_i \dot{X}_j - \frac{RM_P^6}{4\pi^2} [X^i, X^k] [X^k, X^j] - \frac{RM_P^6}{16\pi^2} \theta \gamma^i [X_j, \theta] - \frac{RM_P^6}{16\pi^2} \theta \gamma^j [X_i, \theta] \right), \\
T^{-i} &= \mathbf{STr} \left(\frac{1}{2R} \dot{X}_i (\dot{X}_j)^2 - \frac{RM_P^6}{8\pi^2} \dot{X}_i \sum_{j < k} [X^j, X^k]^2 - \frac{RM_P^6}{4\pi^2} [X^i, X^j] [X^j, X^k] \dot{X}_k \right) \\
&\quad - \mathbf{STr} \left(\frac{RM_P^6}{16\pi^2} \theta_\alpha \dot{X}_k [X_m, \theta_\beta] \right) \{ \gamma^k \delta_{im} + \gamma^i \delta_{mk} - 2\gamma^m \delta_{ki} \}_{\alpha\beta} \\
&\quad - \mathbf{STr} \left(\frac{iR^2 M_P^9}{64\pi^3} \theta_\alpha [X^k, X^l] [X_m, \theta_\beta] \right) \{ \gamma^{[iklm]} + 2\gamma^{[lm]} \delta_{ki} + 4\delta_{ki} \delta_{lm} \}_{\alpha\beta} \\
&\quad + \mathbf{Tr} \left(\frac{iR^2 M_P^9}{64\pi^3} \theta \gamma^{[ki]} \theta \theta \gamma^k \theta \right), \\
T^{--} &= \mathbf{STr} \left(\frac{1}{4R} (\dot{X}^i)^2 (\dot{X}^j)^2 + \frac{RM_P^6}{4\pi^2} \dot{X}^i \dot{X}^j [X^i, X^k] [X^k, X^j] + \frac{RM_P^6}{8\pi^2} (\dot{X}^i)^2 \sum_{j < k} [X^j, X^k]^2 \right. \\
&\quad + \frac{R^3 M_P^{12}}{64\pi^4} [X^i, X^j] [X^j, X^k] [X^k, X^m] [X^m, X^i] - \frac{R^3 M_P^{12}}{64\pi^4} \sum_{i < j} [X^i, X^j]^2 \sum_{k < m} [X^k, X^m]^2 \\
&\quad \left. + \mathcal{O}(\theta^2) + \mathcal{O}(\theta^4) \right). \tag{79}
\end{aligned}$$

To these components, one should now perform the T -duality for the 9 – 11 flip, followed by the rescalings of world-sheet coordinates, background fields and coupling constants. The final result to obtain is the explicit form of the previous components of the stress tensor, this time in matrix string theory (with $i, j \neq 9$):

$$\begin{aligned}
T^{++} &= \frac{1}{2\pi} \left(\frac{\ell_s}{R} \right)^2 \int d\sigma d\tau \mathbf{STr} (\mathbb{1}), \\
T^{+i} &= \frac{1}{2\pi} \left(\frac{\ell_s}{R} \right) \int d\sigma d\tau \mathbf{STr} (\dot{X}_i), \\
T^{+9} &= \frac{1}{2\pi} \left(\frac{\ell_s}{R} \right) \int d\sigma d\tau \mathbf{STr} (g_s \dot{A}), \\
T^{+-} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\frac{1}{2} \dot{X}_i^2 + \frac{1}{2} (DX^i)^2 + \frac{1}{2} g_s^2 \dot{A}^2 - \frac{1}{2g_s^2} \sum_{i < j} [X^i, X^j]^2 \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{g_s} \theta \gamma_i [X^i, \theta] + i \theta \gamma^9 D \theta), \\
T^{ij} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} (\dot{X}_i \dot{X}_j - DX^i DX^j - \frac{1}{g_s^2} [X^i, X^k] [X^k, X^j] \\
& - \frac{1}{2g_s} \theta \gamma^i [X_j, \theta] - \frac{1}{2g_s} \theta \gamma^j [X_i, \theta]), \\
T^{i9} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} (g_s \dot{X}_i \dot{A} + \frac{i}{g_s} [X^i, X^k] DX^k - \frac{i}{2} \theta \gamma^i D\theta - \frac{1}{2g_s} \theta \gamma^9 [X_i, \theta]), \\
T^{99} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} (g_s^2 \dot{A}^2 - (DX^k)^2 - i \theta \gamma^9 D\theta), \\
T^{-i} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(\frac{1}{2} \dot{X}_i (\dot{X}_j)^2 + \frac{1}{2} g_s^2 \dot{X}_i \dot{A}^2 - \frac{1}{2g_s^2} \dot{X}_i \sum_{j < k} [X^j, X^k]^2 + \frac{1}{2} \dot{X}_i (DX^j)^2 \right. \\
& - \frac{1}{g_s^2} [X^i, X^j] [X^j, X^k] \dot{X}_k - DX^i DX^k \dot{X}_k + i [X^i, X^j] DX^j \dot{A} \\
& - \frac{1}{2g_s} \theta_\alpha \dot{X}_k [X_j, \theta_\beta] \{ \gamma^k \delta_{ij} + \gamma^i \delta_{jk} - 2\gamma^j \delta_{ki} \}_{\alpha\beta} - \frac{1}{2} \dot{A} \theta \gamma^9 [X_i, \theta] + i \dot{X}_i \theta \gamma^9 D\theta \\
& - \frac{i}{2} g_s \dot{A} \theta \gamma^i D\theta - \frac{i}{4g_s^2} \theta_\alpha [X^k, X^j] [X^l, \theta_\beta] \{ \gamma^{[ikj]l} + 2\gamma^{[jl]} \delta_{ki} + 4\delta_{ki} \delta_{jl} \}_{\alpha\beta} \\
& - \frac{1}{2g_s} \theta_\alpha DX^k [X^j, \theta_\beta] \{ \gamma^{[ik9j]} + \gamma^{[9j]} \delta_{ki} \}_{\alpha\beta} + \frac{1}{4g_s} \theta_\alpha [X^k, X^j] D\theta_\beta \{ \gamma^{[ikj9]} + 2\gamma^{[j9]} \delta_{ki} \}_{\alpha\beta} \\
& \left. - i DX^i \theta D\theta + \frac{i}{2} \left(\frac{R}{\ell_s} \right) (\theta \gamma^{[ki]} \theta \theta \gamma^k \theta + \theta \gamma^{[9i]} \theta \theta \gamma^9 \theta) \right), \\
T^{-9} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(\frac{1}{2} g_s \dot{A} (\dot{X}^i)^2 + \frac{1}{2} g_s^3 \dot{A}^3 - \frac{1}{2g_s} \dot{A} \sum_{i < j} [X^i, X^j]^2 \right. \\
& - \frac{i}{g_s} DX^i [X^i, X^j] \dot{X}_j - \frac{1}{2} g_s (DX^i)^2 \dot{A} - \frac{1}{2g_s} \dot{X}^i \theta \gamma^9 [X^i, \theta] + \dot{A} \theta \gamma^i [X^i, \theta] \\
& - \frac{i}{2} \dot{X}^i \theta \gamma^i D\theta - \frac{i}{4g_s^2} [X^i, X^j] \theta \gamma^{[9ijk]} [X^k, \theta] + \frac{1}{2g_s} \theta_\alpha DX^i [X^j, \theta_\beta] \{ \gamma^{[ij]} + 2\delta_{ij} \}_{\alpha\beta} \\
& \left. + \frac{i}{2} DX^i \theta \gamma^{[i9]} D\theta + \frac{i}{2} \left(\frac{R}{\ell_s} \right) \theta \gamma^{[i9]} \theta \theta \gamma^i \theta \right), \\
T^{--} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right)^2 \int d\sigma d\tau \mathbf{STr} \left(\frac{1}{4} (\dot{X}^i)^2 (\dot{X}^j)^2 + \frac{1}{2} g_s^2 \dot{A}^2 (\dot{X}^i)^2 + \frac{1}{4} g_s^4 \dot{A}^4 \right. \\
& + \frac{1}{g_s^2} \dot{X}^i \dot{X}^j [X^i, X^k] [X^k, X^j] - 2i \dot{X}^i \dot{A} [X^i, X^k] DX^k + \frac{1}{2} g_s^2 \dot{A}^2 (DX^i)^2 \\
& + \dot{X}^i \dot{X}^j DX^i DX^j + \frac{1}{2g_s^2} (\dot{X}^i)^2 \sum_{j < k} [X^j, X^k]^2 - \frac{1}{2} (\dot{X}^i)^2 (DX^j)^2 + \frac{1}{2} \dot{A}^2 \sum_{i < j} [X^i, X^j]^2 \\
& + \frac{1}{4g_s^4} [X^i, X^j] [X^j, X^k] [X^k, X^m] [X^m, X^i] + \frac{1}{g_s^2} DX^i DX^j [X^i, X^k] [X^k, X^j] \\
& - \frac{1}{4g_s^4} \sum_{i < j} [X^i, X^j]^2 \sum_{k < m} [X^k, X^m]^2 + \frac{1}{2g_s^2} (DX^i)^2 \sum_{j < k} [X^j, X^k]^2 + \frac{1}{4} (DX^i)^2 (DX^j)^2 \\
& \left. + \mathcal{O}(\theta^2) + \mathcal{O}(\theta^4) \right). \tag{80}
\end{aligned}$$

Finally, the free string limit can be taken. The result for the conformal field theory limit of the matrix string stress tensor is:

$$\begin{aligned}
\lim_{g_s \rightarrow 0} T^{++} &= \frac{1}{2\pi} \left(\frac{\ell_s}{R} \right)^2 \int d\sigma d\tau \mathbf{STr} (\mathbf{1}), \\
\lim_{g_s \rightarrow 0} T^{+i} &= \frac{1}{2\pi} \left(\frac{\ell_s}{R} \right) \int d\sigma d\tau \mathbf{STr} (\dot{X}_i), \\
\lim_{g_s \rightarrow 0} T^{+9} &= 0, \\
\lim_{g_s \rightarrow 0} T^{+-} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\frac{1}{2} \dot{X}_i^2 + \frac{1}{2} (\partial X^i)^2 + i\theta \gamma^9 \partial \theta \right), \\
\lim_{g_s \rightarrow 0} T^{ij} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} (\dot{X}_i \dot{X}_j - \partial X^i \partial X^j), \\
\lim_{g_s \rightarrow 0} T^{i9} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{i}{2} \theta \gamma^i \partial \theta \right), \\
\lim_{g_s \rightarrow 0} T^{99} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-(\partial X^i)^2 - i\theta \gamma^9 \partial \theta \right), \\
\lim_{g_s \rightarrow 0} T^{-i} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(\frac{1}{2} \dot{X}_i (\dot{X}_j)^2 + \frac{1}{2} \dot{X}_i (\partial X^j)^2 - \partial X^i \partial X^k \dot{X}_k \right. \\
&\quad \left. + i\dot{X}_i \theta \gamma^9 \partial \theta - i\partial X^i \theta \partial \theta + \mathcal{O} \left(\frac{R}{\ell_s} \right) \right), \\
\lim_{g_s \rightarrow 0} T^{-9} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(-\frac{i}{2} \dot{X}_i \theta \gamma^i \partial \theta + \frac{i}{2} \partial X^i \theta \gamma^{[i9]} \partial \theta + \mathcal{O} \left(\frac{R}{\ell_s} \right) \right), \\
\lim_{g_s \rightarrow 0} T^{--} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right)^2 \int d\sigma d\tau \mathbf{STr} \left(\frac{1}{4} (\dot{X}^i)^2 (\dot{X}^j)^2 + \dot{X}^i \dot{X}^j \partial X^i \partial X^j - \frac{1}{2} (\dot{X}^i)^2 (\partial X^j)^2 \right. \\
&\quad \left. + \frac{1}{4} (\partial X^i)^2 (\partial X^j)^2 + \mathcal{O}(\theta^2) + \mathcal{O}(\theta^4) \right). \tag{81}
\end{aligned}$$

The next terms we look at are the zeroth moments of the components of the Matrix membrane current. These components are:

$$\begin{aligned}
J^{+ij} &= -\frac{iM_P^3}{12\pi} \mathbf{STr} ([X^i, X^j]), \\
J^{+-i} &= \mathbf{STr} \left(\frac{iM_P^3}{12\pi} [X^i, X^j] \dot{X}^j - \frac{RM_P^6}{48\pi^2} \theta [X^i, \theta] + \frac{RM_P^6}{96\pi^2} \theta \gamma^{[ki]} [X^k, \theta] \right), \\
J^{ijk} &= -\mathbf{STr} \left(\frac{iM_P^3}{12\pi} \dot{X}^i [X^j, X^k] + \frac{iM_P^3}{12\pi} \dot{X}^j [X^k, X^i] + \frac{iM_P^3}{12\pi} \dot{X}^k [X^i, X^j] - \frac{RM_P^6}{96\pi^2} \theta \gamma^{[ijkl]} [X^l, \theta] \right), \\
J^{-ij} &= \mathbf{STr} \left(\frac{iM_P^3}{12\pi} \dot{X}^i \dot{X}^k [X^k, X^j] - \frac{iM_P^3}{12\pi} \dot{X}^j \dot{X}^k [X^k, X^i] - \frac{iM_P^3}{24\pi} (\dot{X}^k)^2 [X^i, X^j] \right. \\
&\quad \left. - \frac{iR^2 M_P^9}{96\pi^3} [X^i, X^j] \sum_{k < l} [X^k, X^l]^2 - \frac{iR^2 M_P^9}{48\pi^3} [X^i, X^k] [X^k, X^l] [X^l, X^j] \right) \\
&\quad + \frac{RM_P^6}{96\pi^2} \mathbf{STr} \left(\theta_\alpha \dot{X}^k [X^m, \theta_\beta] \right) \{ \gamma^{[kijm]} + \gamma^{[jmi]} \delta_{ki} - \gamma^{[im]} \delta_{kj} + 2\delta_{jm} \delta_{ki} - 2\delta_{im} \delta_{kj} \}_{\alpha\beta}
\end{aligned}$$

$$\begin{aligned}
& + \frac{iR^2 M_P^9}{64\pi^3} \mathbf{STr} \left(\theta_\alpha [X^k, X^l] [X^m, \theta_\beta] \right) \{ \gamma^{[jkl]} \delta_{mi} - \gamma^{[ikl]} \delta_{mj} + 2\gamma^{[lij]} \delta_{km} + 2\gamma^l \delta_{jk} \delta_{im} \\
& \quad - 2\gamma^l \delta_{ik} \delta_{jm} + 2\gamma^j \delta_{il} \delta_{km} - 2\gamma^i \delta_{jl} \delta_{km} \} \alpha\beta \\
& + \frac{iR^2 M_P^9}{384\pi^3} \mathbf{STr} \left(\theta \gamma^{[kij]} \theta \theta \gamma^k \theta - \theta \gamma^{[ij]} \theta \theta \theta \right). \tag{82}
\end{aligned}$$

To these components we now perform the T -duality for the 9 – 11 flip, followed by the rescalings of world-sheet coordinates, background fields and coupling constants. One obtains the explicit form of the previous components of the membrane current, in matrix string theory (with $i, j, k \neq 9$):

$$\begin{aligned}
J^{+ij} &= \frac{1}{2\pi} \left(\frac{\ell_s}{R} \right) \int d\sigma d\tau \mathbf{STr} \left(-\frac{i}{6g_s} [X^i, X^j] \right), \\
J^{+i9} &= \frac{1}{2\pi} \left(\frac{\ell_s}{R} \right) \int d\sigma d\tau \mathbf{STr} \left(-\frac{1}{6} DX^i \right), \\
J^{+-i} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\frac{i}{6g_s} [X^i, X^j] \dot{X}^j + \frac{1}{6} g_s \dot{A} DX^i - \frac{1}{6g_s} \theta [X^i, \theta] \right. \\
& \quad \left. + \frac{1}{12g_s} \theta \gamma^{[ki]} [X^k, \theta] + \frac{i}{12} \theta \gamma^{[9i]} D\theta \right), \\
J^{+-9} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{1}{6} \dot{X}^i DX^i - \frac{i}{6} \theta D\theta + \frac{1}{12g_s} \theta \gamma^{[i9]} [X^i, \theta] \right), \\
J^{ijk} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{i}{6g_s} \dot{X}^i [X^j, X^k] - \frac{i}{6g_s} \dot{X}^j [X^k, X^i] - \frac{i}{6g_s} \dot{X}^k [X^i, X^j] \right. \\
& \quad \left. + \frac{1}{12g_s} \theta \gamma^{[ijkl]} [X^l, \theta] + \frac{i}{12} \theta \gamma^{[ijk9]} D\theta \right), \\
J^{ij9} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{1}{6} \dot{X}^i DX^j + \frac{1}{6} \dot{X}^j DX^i - \frac{i}{6} \dot{A} [X^i, X^j] + \frac{1}{12g_s} \theta \gamma^{[ij9l]} [X^l, \theta] \right), \\
J^{-ij} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(\frac{i}{6g_s} \dot{X}^i \dot{X}^k [X^k, X^j] - \frac{i}{6g_s} \dot{X}^j \dot{X}^k [X^k, X^i] - \frac{g_s}{6} \dot{A} \dot{X}^i DX^j \right. \\
& \quad + \frac{g_s}{6} \dot{A} \dot{X}^j DX^i - \frac{i}{12g_s} (\dot{X}^k)^2 [X^i, X^j] - \frac{i}{12} g_s \dot{A}^2 [X^i, X^j] - \frac{i}{12g_s^3} [X^i, X^j] \sum_{k<l} [X^k, X^l]^2 \\
& \quad + \frac{i}{12g_s} [X^i, X^j] (DX^k)^2 - \frac{i}{6g_s^3} [X^i, X^k] [X^k, X^l] [X^l, X^j] - \frac{i}{6g_s} DX^i DX^k [X^k, X^j] \\
& \quad + \frac{i}{6g_s} DX^j DX^k [X^k, X^i] + \frac{1}{12g_s} \theta_\alpha \dot{X}^k [X^m, \theta_\beta] \{ \gamma^{[kijm]} + \gamma^{[jlm]} \delta_{ki} - \gamma^{[im]} \delta_{kj} \\
& \quad \quad + 2\delta_{jm} \delta_{ki} - 2\delta_{im} \delta_{kj} \} \alpha\beta \\
& \quad + \frac{1}{12} \dot{A} \theta \gamma^{[9ijm]} [X^m, \theta] + \frac{i}{12} \theta_\alpha \dot{X}^k D\theta_\beta \{ \gamma^{[kij9]} + \gamma^{[j9]} \delta_{ki} - \gamma^{[i9]} \delta_{kj} \} \alpha\beta \\
& \quad + \frac{i}{4g_s^2} \theta_\alpha [X^k, X^l] [X^m, \theta_\beta] \{ \gamma^{[jkl]} \delta_{mi} - \gamma^{[ikl]} \delta_{mj} + 2\gamma^{[lij]} \delta_{km} + 2\gamma^l \delta_{jk} \delta_{im} \\
& \quad \quad - 2\gamma^l \delta_{ik} \delta_{jm} + 2\gamma^j \delta_{il} \delta_{km} - 2\gamma^i \delta_{jl} \delta_{km} \} \alpha\beta
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2g_s} \theta_\alpha D X^k [X^m, \theta_\beta] \{ \gamma^{[jk9]} \delta_{mi} - \gamma^{[ik9]} \delta_{mj} + \gamma^{[9ij]} \delta_{km} + \gamma^9 \delta_{jk} \delta_{im} - \gamma^9 \delta_{ik} \delta_{jm} \}_{\alpha\beta} \\
& - \frac{i}{2} \theta_\alpha D X^l D \theta_\beta \{ \gamma^{[lij]} + \gamma^j \delta_{il} - \gamma^i \delta_{jl} \}_{\alpha\beta} \\
& + \frac{i}{12} \left(\frac{R}{\ell_s} \right) \left(\theta \gamma^{[kij]} \theta \theta \gamma^k \theta + \theta \gamma^{[9ij]} \theta \theta \gamma^9 \theta - \theta \gamma^{[ij]} \theta \theta \theta \right), \\
J^{-i9} = & \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(\frac{1}{6} \dot{X}^i \dot{X}^k D X^k - \frac{i}{6} \dot{A} \dot{X}^k [X^k, X^i] + \frac{1}{12} g_s^2 \dot{A}^2 D X^i \right. \\
& - \frac{1}{12} (\dot{X}^k)^2 D X^i - \frac{1}{12 g_s^2} D X^i \sum_{k < l} [X^k, X^l]^2 - \frac{1}{12} D X^i (D X^k)^2 \\
& - \frac{1}{6 g_s^2} [X^i, X^k] [X^k, X^l] D X^l + \frac{1}{12 g_s} \theta_\alpha \dot{X}^k [X^m, \theta_\beta] \{ \gamma^{[ki9m]} + \gamma^{[9m]} \delta_{ki} \}_{\alpha\beta} \\
& - \frac{1}{12} \dot{A} \theta_\alpha [X^m, \theta_\beta] \{ \gamma^{[im]} + 2 \delta_{im} \}_{\alpha\beta} + \frac{i}{6} \dot{X}^i \theta D \theta - \frac{i}{12} \dot{A} \theta \gamma^{[i9]} D \theta \\
& + \frac{i}{4 g_s^2} \theta_\alpha [X^k, X^l] [X^m, \theta_\beta] \{ \gamma^{[9kl]} \delta_{mi} + 2 \gamma^{[li9]} \delta_{km} + 2 \gamma^9 \delta_{il} \delta_{km} \}_{\alpha\beta} \\
& - \frac{1}{2 g_s} D X^k \theta \gamma^k [X^i, \theta] - \frac{1}{2 g_s} D X^k \theta \gamma^i [X^k, \theta] \\
& + \frac{i}{4 g_s} \theta_\alpha [X^k, X^l] D \theta_\beta \{ \gamma^{[ikl]} + 2 \gamma^l \delta_{ik} \}_{\alpha\beta} - i D X^i \theta \gamma^9 D \theta \\
& \left. + \frac{i}{12} \left(\frac{R}{\ell_s} \right) \left(\theta \gamma^{[ki9]} \theta \theta \gamma^k \theta - \theta \gamma^{[i9]} \theta \theta \theta \right) \right). \tag{83}
\end{aligned}$$

The free string limit can now be taken. The result for the conformal field theory limit of the matrix string membrane current is:

$$\begin{aligned}
\lim_{g_s \rightarrow 0} J^{+ij} &= 0, \\
\lim_{g_s \rightarrow 0} J^{+i9} &= \frac{1}{2\pi} \left(\frac{\ell_s}{R} \right) \int d\sigma d\tau \mathbf{STr} \left(-\frac{1}{6} \partial X^i \right), \\
\lim_{g_s \rightarrow 0} J^{+-i} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\frac{i}{12} \theta \gamma^{[9i]} \partial \theta \right), \\
\lim_{g_s \rightarrow 0} J^{+-9} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{1}{6} \dot{X}^i \partial X^i - \frac{i}{6} \theta \partial \theta \right), \\
\lim_{g_s \rightarrow 0} J^{ijk} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\frac{i}{12} \theta \gamma^{[ijk9]} \partial \theta \right), \\
\lim_{g_s \rightarrow 0} J^{ij9} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{1}{6} \dot{X}^i \partial X^j + \frac{1}{6} \dot{X}^j \partial X^i \right), \\
\lim_{g_s \rightarrow 0} J^{-ij} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(\frac{i}{12} \theta_\alpha \dot{X}^k \partial \theta_\beta \{ \gamma^{[kij9]} + \gamma^{[j9]} \delta_{ki} - \gamma^{[i9]} \delta_{kj} \}_{\alpha\beta} \right. \\
& \quad \left. - \frac{i}{2} \theta_\alpha \partial X^l \partial \theta_\beta \{ \gamma^{[lij]} + \gamma^j \delta_{il} - \gamma^i \delta_{jl} \}_{\alpha\beta} + \mathcal{O} \left(\frac{R}{\ell_s} \right) \right), \\
\lim_{g_s \rightarrow 0} J^{-i9} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(\frac{1}{6} \dot{X}^i \dot{X}^k \partial X^k - \frac{1}{12} (\dot{X}^k)^2 \partial X^i - \frac{1}{12} \partial X^i (\partial X^k)^2 \right)
\end{aligned}$$

$$+\frac{i}{6}\dot{X}^i\theta\partial\theta-i\partial X^i\theta\gamma^9\partial\theta+\mathcal{O}\left(\frac{R}{\ell_s}\right). \quad (84)$$

Next, we look at the zeroth moments of the components of the Matrix 5-brane current. Explicitly, these components are:

$$\begin{aligned} M^{+-ijkl} &= \mathbf{STr}\left(-\frac{RM_P^6}{48\pi^2}[X^i, X^j][X^k, X^l]-\frac{RM_P^6}{48\pi^2}[X^i, X^k][X^l, X^j]\right. \\ &\quad \left.-\frac{RM_P^6}{48\pi^2}[X^i, X^l][X^j, X^k]+\frac{RM_P^6}{48\pi^2}\theta\gamma^{[jkl}[\dot{X}^i], \theta]\right), \\ M^{-ijklm} &= \mathbf{STr}\left(-\frac{5RM_P^6}{16\pi^2}\dot{X}^i[X^j, X^k][X^l, X^m]-\frac{5RM_P^6}{48\pi^2}\theta\dot{X}^{[i}\gamma^{jkl}[\dot{X}^m], \theta]\right. \\ &\quad \left.-\frac{5iR^2M_P^9}{192\pi^3}\theta[X^i, X^j]\gamma^{klm}\gamma^n[X^n, \theta]\right). \end{aligned} \quad (85)$$

To these components one now performs the T -duality for the 9 – 11 flip, followed by the rescalings of world-sheet coordinates, background fields and coupling constants. We then obtain the explicit form of the previous components of the matrix string theory 5-brane current (with $i, j, k, l, m \neq 9$):

$$\begin{aligned} M^{+-ijkl} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(-\frac{1}{12g_s^2}[X^i, X^j][X^k, X^l] - \frac{1}{12g_s^2}[X^i, X^k][X^l, X^j] \right. \\ &\quad \left. - \frac{1}{12g_s^2}[X^i, X^l][X^j, X^k] + \frac{1}{6g_s}\theta\gamma^{[jkl}[\dot{X}^i], \theta] \right), \\ M^{+-ijk9} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\frac{i}{2g_s}DX^k[X^i, X^j] - \frac{i}{2g_s}DX^j[X^i, X^k] + \frac{i}{2g_s}DX^i[X^j, X^k] \right. \\ &\quad \left. + \frac{i}{6}\theta\gamma^{[ikj]}D\theta - \frac{1}{6g_s} \left(\theta\gamma^{9[kj}[\dot{X}^i], \theta] + \theta\gamma^{[i9]j}[\dot{X}^k], \theta] + \theta\gamma^{[ik]9}[\dot{X}^j], \theta] \right) \right), \\ M^{-ijklm} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(-\frac{5}{4g_s^2}\dot{X}^{[i}[X^j, X^k][X^l, X^m] - \frac{5}{6g_s}\theta\dot{X}^{[i}\gamma^{jkl}[\dot{X}^m], \theta] \right. \\ &\quad \left. - \frac{5i}{12g_s^2}\theta[X^i, X^j]\gamma^{klm}\gamma^n[X^n, \theta] + \frac{5}{12g_s}\theta[X^i, X^j]\gamma^{klm}\gamma^9D\theta \right), \\ M^{-ijkl9} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(\frac{5i}{g_s}\dot{X}^{[i}[X^j, X^k]DX^l] + \frac{5}{4g_s}\dot{X}^{[i}[X^j, X^k][X^l, X^k] \right. \\ &\quad \left. - \frac{5i}{6}\theta\dot{X}^{[i}\gamma^{jkl]}D\theta + \frac{5}{6}\dot{X}^{[i}\theta\gamma^{jkl}[\dot{X}^k], \theta] + \frac{5}{2g_s}\theta\dot{X}^{[i}\gamma^{9]kl}[\dot{X}^j], \theta] \right. \\ &\quad \left. - \frac{5i}{4g_s^2}\theta[X^i, X^j]\gamma^{kl9}\gamma^n[X^n, \theta] + \frac{5}{4g_s}\theta[X^i, X^j]\gamma^{kl9}\gamma^9D\theta \right. \\ &\quad \left. - \frac{5}{6g_s}\theta DX^{[i}\gamma^{jkl]}\gamma^n[X^n, \theta] - \frac{5i}{6}\theta DX^{[i}\gamma^{jkl]}\gamma^9D\theta \right). \end{aligned} \quad (86)$$

One can now take the free string limit. The result for the conformal field theory limit of the matrix string 5-brane current is:

$$\begin{aligned}
\lim_{g_s \rightarrow 0} M^{+-ijkl} &= 0, \\
\lim_{g_s \rightarrow 0} M^{+-ijk9} &= \frac{1}{2\pi} \int d\sigma d\tau \mathbf{STr} \left(\frac{i}{6} \theta \gamma^{[ikj]} \partial \theta \right), \\
\lim_{g_s \rightarrow 0} M^{-ijklm} &= 0, \\
\lim_{g_s \rightarrow 0} M^{-ijkl9} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(-\frac{5i}{6} \theta \dot{X}^{[i} \gamma^{jkl]} \partial \theta - \frac{5i}{6} \theta \partial X^{[i} \gamma^{jlk]} \gamma^9 \partial \theta \right). \quad (87)
\end{aligned}$$

Finally, we look at are the zeroth moments of the components of the Matrix 6-brane current (related to nontrivial 11-dimensional background metrics), which are given explicitly by:

$$\begin{aligned}
S^{+ijklmn} &= -\frac{iR^2 M_P^9}{8\pi^3} \mathbf{STr} \left([X^i, X^j][X^k, X^l][X^m, X^n] \right), \\
S^{ijklmnp} &= \mathbf{STr} \left(-\frac{7iR^2 M_P^9}{8\pi^3} [X^i, X^j][X^k, X^l][X^m, X^n] \dot{X}^p + \mathcal{O}(\theta^2, \theta^4) \right). \quad (88)
\end{aligned}$$

To these components we now perform T -duality for the 9 – 11 flip, followed by the rescalings of world-sheet coordinates, background fields and coupling constants. We obtain the explicit form of the previous components of the matrix string theory 6-brane current (with $i, j, k, l, m, n \neq 9$):

$$\begin{aligned}
S^{+ijklmn} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(-\frac{i}{g_s^3} [X^i, X^j][X^k, X^l][X^m, X^n] \right), \\
S^{+ijklm9} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right) \int d\sigma d\tau \mathbf{STr} \left(-\frac{6}{g_s^2} [X^i, X^j][X^k, X^l] D X^m \right), \\
S^{ijklmnp} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right)^2 \int d\sigma d\tau \mathbf{STr} \left(-\frac{7i}{g_s^3} [X^i, X^j][X^k, X^l][X^m, X^n] \dot{X}^p + \mathcal{O}(\theta^2, \theta^4) \right), \\
S^{ijklmn9} &= \frac{1}{2\pi} \left(\frac{R}{\ell_s} \right)^2 \int d\sigma d\tau \mathbf{STr} \left(-\frac{7i}{g_s^2} [X^i, X^j][X^k, X^l][X^m, X^n] \dot{A} \right. \\
&\quad \left. - \frac{42}{g_s^2} D X^{[j} [X^k, X^l][X^m, X^n] \dot{X}^{i]} + \mathcal{O}(\theta^2, \theta^4) \right). \quad (89)
\end{aligned}$$

One can now take the free string limit. The result for the conformal field theory limit of the matrix string 6-brane current is:

$$\begin{aligned}
\lim_{g_s \rightarrow 0} S^{+ijklmn} &= 0, \\
\lim_{g_s \rightarrow 0} S^{+ijklm9} &= 0, \\
\lim_{g_s \rightarrow 0} S^{ijklmnp} &= 0, \\
\lim_{g_s \rightarrow 0} S^{ijklmn9} &= 0. \quad (90)
\end{aligned}$$

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